Year 10 Knowledge Organiser

The Bishops' Blue Coat

Church of England High School

Contents Page

Percentages

What do I need to be able to do?

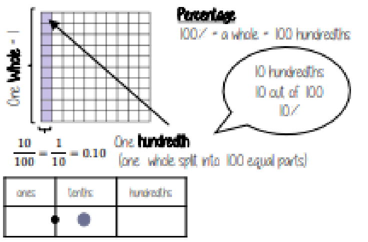
By the end of this unit you should be able to:

- Use FDP equivalence
- Calculate percentage increase and decrease
- Express percentage change
- Solve reverse percentage problems
- Solve percentage problems (calculator and non calculator problems)

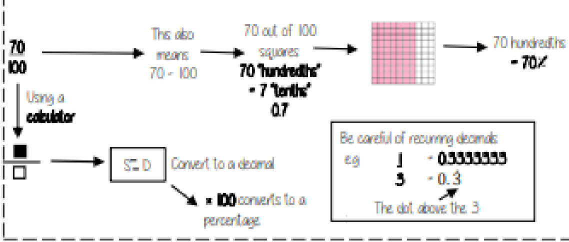
Keywords

Percent: parts per 100 – written using the % symbol
Decimal: a number in our base 10 number system. Numbers to the right of the decimal place are called decimals
Fraction: a fraction represents how many parts of a whole value you have
Equivalent: of equal value
Reduce: to make smaller in value
Growth: to increase/ to grow
Integer: whole number, can be positive, negative or zero
Invest: use money with the goal of it increasing in value over time (usually in a bank)
Multiplier: the number you are multiplying by
Profit: the income take away any expenses/ costs

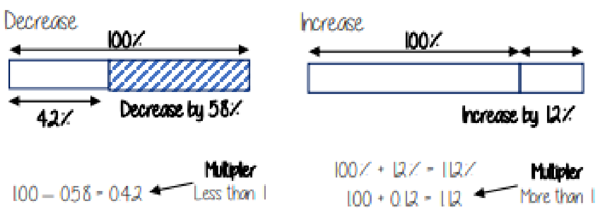
FDP Equivalence (R)



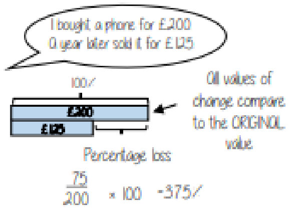
Converting FDP (R)



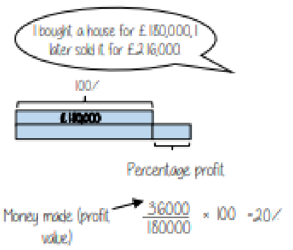
Percentage Increase/ Decrease (R)



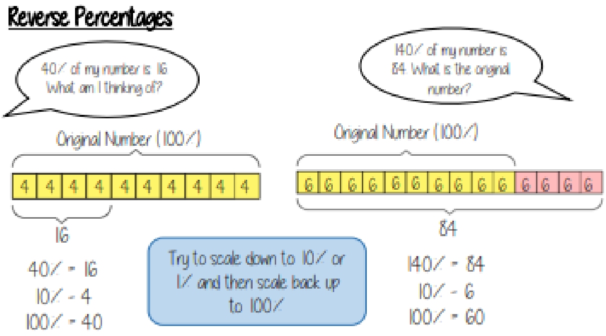
Percentage change (R)



$$\frac{\text{Difference in values}}{\text{Original value}} \times 100$$



Reverse Percentages



Joanne sees this special offer in a shop.

Special Offer

Microwave	£189
Toaster	£25

Buy both items and receive a 4% discount

Joanne buys both items.

How much does she pay?

Ratios and fractions

What do I need to be able to do?

By the end of this unit you should be able to:

- Compare quantities using ratio
- Link ratios and fractions and make comparisons
- Share in a given ratio
- Link Ratio and scales and graphs
- Solve problems with currency conversions
- Solve 'best buy' problems
- Combine ratios

Keywords

Ratio: a statement of how two numbers compare
Equivalent: of equal value
Proportion: a statement that links two ratios
Integer: whole number, can be positive, negative or zero.
Fraction: represents how many parts of a whole
Denominator: the number below the line on a fraction. The number represent the total number of parts
Numerator: the number above the line on a fraction. The top number. Represents how many parts are taken
Origin: (0,0) on a graph. The point the two axes cross
Gradient: The steepness of a line

Compare with ratio (R)

"For every dog there are 2 cats"

Dogs: Cats
1:2

The ratio has to be written in the same order as the information is given.
e.g. 2:1 would represent 2 dogs for every 1 cat

Units have the be of the same value to compare ratios

Ratios and fraction (R)

Trees: Flowers
3 : 7

Trees / Flowers — Ratio

Fraction of trees

Number of parts of in group / Total number of parts = $\frac{3}{10}$ — Fraction

Sharing a whole into a given ratio (R)

James and Lucy share £350 in the ratio 3:4
Work out how much each person earns

Model the Question

James Lucy
3 : 4

James / Lucy — £350

Find the value of one part

Whole: £350
7 parts to share between (3 James, 4 Lucy)

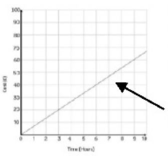
£350 ÷ 7 = £50
□ = one part = £50

Put back into the question

James Lucy
x 50 3 : 4 x 50
£150:£200

James = 3 x £50 = £150
Lucy = 4 x £50 = £200
£350

Ratio and graphs (R)



Graphs with a constant ratio are directly proportional

- Form a straight line
- Pass through (0,0)

The gradient is the constant ratio

Ratio and scale (R)

A picture of a car is drawn with a scale of 1:30

The car image is 10cm

Image : Real life
1cm : 30cm
x 10 ↓ 10cm 300cm ↓ x 10

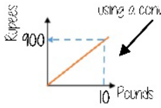
Conversion between currencies (R)

£1 = 90 Rupees ← Currency is directly proportional

"For every £1 I have 90 Rupees"

£1 = 90 Rupees
x 10 ↓ ↓ x 10
£10 = 900 Rupees

Currency can be converted using a conversion graph



Convert 630 Rupees into Pounds

630 ÷ 90 = 7

£1 = 90 Rupees
x 7 ↓ ↓ x 7
£7 = 630 Rupees

Ratios in 1:n and n:1

This is asking you to cancel down until the part indicated represents 1

Show the ratio 4:20 in the ratio of 1:n

4 : 20
1 : 5

The question states that this part has to be 1 unit therefore Divide by 4

This side has to be divided by 4 too – to keep in proportion

The n part does not have to be an integer for this type of question

Best buys

	4 pens costs £2.60	10 pens costs £6.00
"1 pen costs…"	£2.60 ÷ 4 = **£0.65**	£6.00 ÷ 10 = **£0.60**
"1 pound buys…"	4 ÷ 2.60 = **1.54 pens**	10 ÷ 6 = **1.67 pens**

You could work out how much 40 pens are and then compare

Compare the solution in the context of the question

The best value has the lowest cost "per pen"

The best value means £1 buys you more pens

Combining ratios

	B	R	G
The ratio of Blue counters to Red counters is 5:3	5	3	
The ratio of Red counters to Green counters is 2:1		2	1
Ratio of Blue to Red to Green	10	6	3

Use equivalent ratios to allow comparison of the group that is common to both statements

Lowest common multiple of the ratio both statements share

THE BIG QUESTION

Four teachers are planning school trips.
The table shows the number of students and the number of teachers planned to go on the trip.

	Karting	Museum	Theme Park	University
Number of students	140	221	342	159
Number of teachers	8	12	19	9

For every 18 students there must be at least 1 teacher.
Which trips have planned to bring enough teachers?

Karting, Theme Park, University

Money

What do I need to be able to do?

By the end of this unit you should be able to:

- Solve problems with bills and bank statements
- Calculate simple interest
- Calculate compound interest
- Calculate wages and taxes
- Solve problems with exchange rates
- Solve unit pricing problems

Keywords

Credit: money being placed into a bank account
Debit: money that leaves a bank account
Balance: the amount of money in a bank account
Expense: a cost/ outgoing
Deposit: an initial payment (often a way of securing an item you will later pay for)
Multiplier: a number you are multiplying by (Multiplier more than 1 - increasing, less than 1 - decreasing)
Per Annum: each year
Currency: the type of money a country uses
Unitary: one – the cost of one

Bills and Bank Statements

Bills – tell you the amount items cost and can show how much money you need to pay

Some can include a total
Look for different units
(Is it in pence or pounds)

Menu	Price
Milk	89p
Tea	£1.50

Bank Statements

Bank statement can have negative balances if the money spent is higher than the money coming into the account

Date	Description	Credit	Debit	Balance
1st Sept	Salary	£1500		£1500
1st Sept	Mortgage		£600	£900
25th Sept	Bday Money	£15		£915

Simple Interest

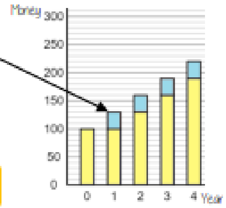
For each year of investment the interest remains the same

$$\frac{\text{Principal amount} \times \text{Interest Rate} \times \text{Years}}{100}$$

Principal amount is the amount invested in the account
e.g. Invest £100 at 30% simple interest for 4 years

$\frac{100 \times 30 \times 4}{100} = £120$

This account earned **£120** interest. At the end of year 4 they have **£220**



Compound Interest

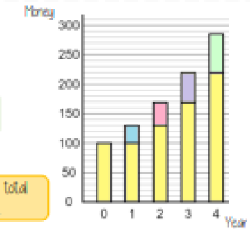
Interest is added to the current value of investment at the end of each year so the next year's interest is greater

$$\text{Principal amount} \times \text{Multiplier}^{\text{Years}}$$

e.g. Invest £100 at 30% compound interest for 4 years

$100 \times 1.3^4 = £285.61$

This account has **£285.61** in total at the end of the 4 years



Value Added Tax (VAT)

VAT is payable to the government by a business. In the UK VAT is 20% and added to items that are bought

Essential items such as food do not include VAT

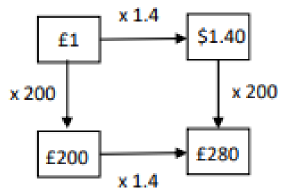
Wages and Taxes

Salaries fall into tax brackets – which means they pay this much each month from their salary

Taxable Income	Tax Rate
£12 501 to £50 000	20%
£50 001 to £150 000	40%
over £150 000	45%

Over time:
Time and a half – means 1.5 times their hourly rate
Double – 2 times their hourly rate

Exchange Rates



When making estimates it is also useful to use estimates to check if our solution is reasonable

Use inverse operations to reverse the exchange process

Common Currencies		
United Kingdom	£	Pounds
United States of America	$	Dollars
Europe	€	Euros

Unit Pricing

4 Oranges	5 cupcakes
£1	£1.20

4 = £1.00
÷ 2
2 = £0.50
÷ 2
1 = £0.25

5 = £1.20
÷ 5
1 = £0.20

Cost per Unit

To calculate unit per cost you divide by the cost

Cupcakes are the best value as one item has the cheapest value

There is a directly proportional relationship between the cost and number of units

Ben works 42 hours each week.
His weekly wage is £453.60

Next year Ben will be paid an extra 65p per hour.

How much will Ben's weekly wage be next year?

THE BIG QUESTION

Non calculator Methods

What do I need to be able to do?

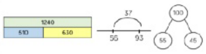
By the end of this unit you should be able to:

- Use mental/written methods for the four number operations
- Use four operations for fractions
- Write exact answers
- Round to decimal places and significant figures
- Estimate solutions
- Understand limits of accuracy
- Understand financial maths

Keywords

Truncate: to shorten, to shorten a number (no rounding), to shorten a shape (remove a part of the shape)
Round: making a number simpler, but keeping its place value close the what it originally was
Credit: money that goes into a bank account
Debit: money that leaves a bank account
Profit: the amount of money after income – costs
Tax: money that the government collects based on income, sales and other activities
Balance: The amount of money in a bank account
Overestimate: Rounding up – gives a solution higher than the actual value
Underestimate: Rounding down – gives a solution lower than the actual value

Addition/ Subtraction



Modelling methods for addition/ subtraction

- Bar models
- Number lines
- Part/ Whole diagrams

Addition is commutative

6 + 3 = 3 + 6

The order of addition does not change the result

Subtraction the order has to stay the same

360 – 147 = 360 – 100 – 40 – 7

- Number lines help for addition and subtraction
- Working in 10's first aids mental addition/ subtraction
- Show your relationships by writing fact families

Formal written methods (R)

	H	T	O
	1	8	7
+	5	4	2

	H	T	O
	4	2	7
–	2	4	9

Remember the place value of each column
You may need to move 10 ones to the ones column to be able to subtract

Decimals have the same methods remember to align the place value

Division methods

Short division

$3584 \div 7 = 512$

$7 \overline{)3\ {}^{3}5\ 8\ {}^{1}4}$ = 5 1 2

Complex division

$\div 24 = \div 6 \div 4$

Break up the divisor using factors

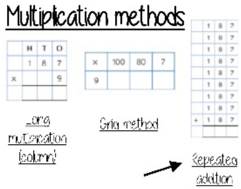
Division with decimals

The placeholder in division methods is essential – the decimal lines up on the dividend and the quotient

$24 \div 0.02 \longrightarrow 24 \div 0.2 \longrightarrow 240 \div 2$

All give the same solution as represent the same proportion
Multiply the values in proportion until the divisor becomes an integer (R)

Multiplication methods



Long multiplication (column)

Grid method

Repeated addition

Less effective method especially for bigger multiplication

Multiplication with decimals (R)

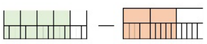
Perform multiplications as integers
e.g. 0.2 x 0.3 ⟶ 2 x 3

Make **adjustments** to your answer to match the question: 0.2 x 10 = 2
0.3 x 10 = 3
Therefore 6 ÷ 100 = 0.06

Four operations with fractions

Addition and Subtraction

$\frac{4}{5} - \frac{2}{3}$

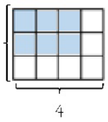


$\frac{12}{15} - \frac{10}{15} = \frac{2}{15}$

Multiplication

$\frac{3}{4} \times \frac{2}{3}$

$= \frac{6}{12} = \frac{1}{2}$



Division (R)

$\frac{2}{5} \div \frac{3}{4}$

$\frac{2}{5} \times \frac{4}{3}$

Multiplying by a reciprocal gives the same outcome

$= \frac{8}{15}$

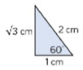
Exact Values

Leave in terms of π

$= \frac{120}{360} \times 36\pi$

$= \frac{1}{3} \times 36\pi = 12\pi$

Leave as a surd



$\tan 30 = \frac{1}{\sqrt{3}}$

Estimation (R)

Round to 1 significant figure to estimate

$21.4 \times 3.1 \approx 20 \times 3 \approx 60$

The equal sign changes to show it is an estimation

This is an **underestimate** because both values were rounded down

It is good to check all calculations with an estimate in all aspects of maths – it helps you identify calculation errors

Limits of accuracy

A width w has been rounded to 6.4cm correct to 1dp

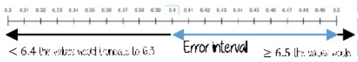
< 6.35 the value would round to 6.3 | Error interval | ≥ 6.45 the values would round to 6.5

The error interval

$6.35 \leq w < 6.45$

Any value within these limits would round to 6.4 to 1dp

A width w has been truncated to 6.4cm correct to 1dp



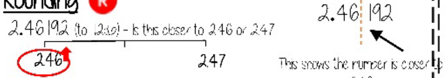
< 6.4 the values would truncate to 6.3 | Error interval | ≥ 6.5 the values would truncate to 6.5

$6.4 \leq w < 6.5$

Any value within these limits would truncate to 6.4 to 1dp

Rounding (R)

2.46192 (to 2dp) – Is this closer to 2.46 or 2.47



2.46 | 192

This shows the number is closer to 2.46

Significant Figures

370 to 1 significant figure is 400
37 to 1 significant figure is 40
3.7 to 1 significant figure is 4
0.37 to 1 significant figure is 0.4
0.000000037 to 1 significant figure is 0.00000004

SF: Round to the first nonzero number

THE BIG QUESTION

(a) Work out $\frac{3}{4} - \frac{7}{10}$

(b) Work out $2\frac{1}{3} \times \frac{3}{5}$

Give your answer as a mixed number in its simplest form

$\frac{5}{2}$ 1

Recurring decimals

You might think that a decimal is just a decimal. But oh no — things get a lot more juicy than that...

Recurring or Terminating...

1) Recurring decimals have a pattern of numbers which repeats forever, e.g. $\frac{1}{3}$ is the decimal 0.333333... Note, it doesn't have to be a single digit that repeats. You could have, for instance: 0.143143143...
2) The repeating part is usually marked with dots or a bar on top of the number. If there's one dot, then only one digit is repeated. If there are two dots, then everything from the first dot to the second dot is the repeating bit. E.g. $0.2\dot{5} = 0.2555555...$, $0.\dot{2}\dot{5} = 0.25252525...$, $0.\dot{2}5\dot{5} = 0.255255255...$
3) Terminating decimals are finite (they come to an end), e.g $\frac{1}{20}$ is the decimal 0.05.

The denominator (bottom number) of a fraction in its simplest form tells you if it converts to a recurring or terminating decimal. Fractions where the denominator has prime factors of only 2 or 5 will give terminating decimals. All other fractions will give recurring decimals. **H**

	Only prime factors: 2 and 5				Also other prime factors			
Fraction	$\frac{1}{5}$	$\frac{1}{125}$	$\frac{1}{2}$	$\frac{1}{20}$	$\frac{1}{7}$	$\frac{1}{35}$	$\frac{1}{3}$	$\frac{1}{6}$
Equivalent Decimal	0.2	0.008	0.5	0.05	$0.\dot{1}4285\dot{7}$	$0.0\dot{2}8571\dot{4}$	$0.\dot{3}$	$0.1\dot{6}$
	Terminating decimals				Recurring decimals			

For prime factors, see p.3

Converting terminating decimals into fractions was covered on the previous page.
Converting recurring decimals is quite a bit harder — but you'll be OK once you've learnt the method...

Recurring Decimals into Fractions

1) Basic Ones

Turning a recurring decimal into a fraction uses a really clever trick. Just watch this...

EXAMPLE: Write $0.\dot{2}3\dot{4}$ as a fraction.

1) Name your decimal — I've called it r. — Let $r = 0.\dot{2}3\dot{4}$
2) Multiply r by a power of ten to move it past the decimal point by one full repeated lump — here that's 1000: — $1000r = 234.\dot{2}3\dot{4}$
3) Now you can subtract to get rid of the decimal part:

$$\begin{aligned} 1000r &= 234.\dot{2}3\dot{4} \\ -\quad r &= 0.\dot{2}3\dot{4} \\ \hline 999r &= 234 \end{aligned}$$

4) Then just divide to leave r, and cancel if possible: — $r = \frac{234}{999} = \frac{26}{111}$

2) The Trickier Type **H**

If the recurring bit doesn't come right after the decimal point, things are slightly trickier — but only slightly.

EXAMPLE: Write $0.1\dot{6}$ as a fraction.

1) Name your decimal. — Let $r = 0.1\dot{6}$
2) Multiply r by a power of ten to move the non-repeating part past the decimal point. — $10r = 1.\dot{6}$
3) Now multiply again to move one full repeated lump past the decimal point. — $100r = 16.\dot{6}$
4) Subtract to get rid of the decimal part:

$$\begin{array}{r} 100r = 16.\dot{6} \\ -\quad 10r = 1.\dot{6} \\ \hline 90r = 15 \end{array}$$

5) Divide to leave r, and cancel if possible: $r = \frac{15}{90} = \frac{1}{6}$

Fractions into Recurring Decimals

You might find this cropping up in your exam too — and if they're being really unpleasant, they'll stick it in a non-calculator paper.

EXAMPLE: Write $\frac{8}{33}$ as a recurring decimal.

There are two ways you can do this:

1 Find an equivalent fraction with all nines on the bottom. The number on the top will tell you the recurring part.

$$\frac{8}{33} \overset{\times 3}{=} \frac{24}{99}$$

$$\frac{24}{99} = 0.\dot{2}\dot{4}$$

Watch out — the number of nines on the bottom tells you the number of digits in the recurring part. E.g. $\frac{24}{99} = 0.\dot{2}\dot{4}$, but $\frac{24}{999} = 0.\dot{0}2\dot{4}$

2 Remember, $\frac{8}{33}$ means $8 \div 33$, so you could just do the division: (This is OK if you're allowed your calculator, but a bit tricky if not... you can use short or long division if you're feeling bold, but I recommend sticking with method 1 instead.)

$$33\overline{)8.0^{14}0^{8}0^{14}0^{8}00} = 0.2\ 4\ 2\ 4...$$

$$\frac{8}{33} = 0.\dot{2}\dot{4}$$

THE BIG QUESTION

Convert $0.3\dot{4}$ to a fraction.
Give your answer in its simplest form.

Circle the largest number.

$1.\dot{8}\dot{5}$ $\quad 1.8\dot{5}$ $\quad 1.85$ $\quad 1.8$

Indices

What do I need to be able to do?

By the end of this unit you should be able to:

- Identify square and cube numbers
- Calculate higher powers and roots
- Understand powers of 10 and standard form
- Know the addition and subtraction rule for indices
- Understand power zero and negative indices
- Calculate with numbers in standard form

Keywords

Standard (index) Form: A system of writing very big or very small numbers
Commutative: an operation is commutative if changing the order does not change the result.
Base: The number that gets multiplied by a power
Power: The exponent – or the number that tells you how many times to use the number in multiplication
Exponent: The power – or the number that tells you how many times to use the number in multiplication
Indices: The power or the exponent.
Negative: A value below zero.
Coefficient: The number used to multiply a variable

Square and cube numbers

R

Square numbers

1, 4, 9, 16…

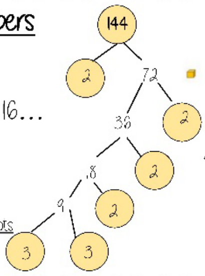
144 = 2 x 2 x 2 x 2 x 3 x 3

2 x 2 x 3 x 2 x 2 x 3

12 x 12

Prime factors can find square roots

$\sqrt{144} = 12$



Cube numbers

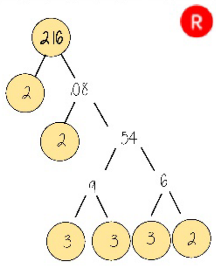
1, 8, 27, 64, 125…

216 = 2 x 2 x 2 x 3 x 3 x 3

2 x 3 x 2 x 3 x 2 x 3

6 x 6 x 6

$\sqrt[3]{216} = 6$



Higher powers and roots

x^n — n – power (number of times multiplied by itself)

x – the base number.

$\sqrt[n]{x}$ — Finding the nth root of any value

Other mental strategies for square roots

$\sqrt{810000} = \sqrt{81} \times \sqrt{10000}$

$= 9 \times 100$

$= 900$

Standard form

R

$A \times 10^n$

Any number between 1 and less than 10

Any integer

0.001
$1 \times \frac{1}{1000}$
1×10^{-3}

10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
10^1	10^0	10^{-1}	10^{-2}	10^{-3}
10	1	0.1	0.01	0.001

Any value to the power 0 always = 1

Example

3.2×10^4

$= 3.2 \times 10 \times 10 \times 10 \times 10$

$= 32000$

Non-example

0.8×10^4

$5.3 \times 10^{0.7}$

Negative powers do not indicate negative solutions

Numbers in standard form with negative powers will be less than 1

$3.2 \times 10^{-4} = 3.2 \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} \times \frac{1}{10} = 0.00032$

Addition/ Subtraction Laws

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

Zero and negative indices

$x^0 = 1$

Any number divided by itself = 1

$\frac{a^6}{a^6} = a^6 \div a^6$

$= a^{6-6} = a^0 = 1$

Negative indices do not indicate negative solutions

$2^2 = 4$
$2^1 = 2$
$2^0 = 1$
$2^{-1} = \frac{1}{2}$
$2^{-2} = \frac{1}{4}$

Looking at the sequence can help to understand negative powers

Powers of powers

$(x^a)^b = x^{ab}$

$(2^3)^4 = 2^3 \times 2^3 \times 2^3 \times 2^3$

The same base and power is repeated Use the addition law for indices

$(2^3)^4 = 2^{12}$ ← $a \times b = 3 \times 4 = 12$

NOTICE the difference

$(2x^3)^4 = 2x^3 \times 2x^3 \times 2x^3 \times 2x^3$

The addition law applies ONLY to the powers. The integers still need to be multiplied

$(2x^3)^4 = 16x^{12}$

Standard form calculations

R

Addition and Subtraction

Tip: Convert into ordinary numbers first and back to standard from at the end

$6 \times 10^5 + 8 \times 10^5$

Method 1

$= 600000 + 800000$
$= 1400000$
$= 1.4 \times 10^6$

This is not the final answer

Method 2

$= (6 + 8) \times 10^5$
$= 14 \times 10^5$
$= 1.4 \times 10 \times 10^5$
$= 1.4 \times 10^6$

Multiplication and division

$\frac{1.5 \times 10^5}{0.3 \times 10^3}$

Division questions can look like this

$(1.5 \times 10^5) \div (0.3 \times 10^3)$

$1.5 \div 0.3 \times 10^5 \div 10^3$

$= 5 \times 10^2$

For multiplication and division you can look at the values for A and the powers of 10 as two separate calculations

THE BIG QUESTION

(a) Simplify $9p^3 \times 2p^{-2}$ (1)

(b) Simplify $(5x^3y^2)^3$ (2)

(c) $p^3 \times p^5 = p^{12} \times p^y$

Find the value of y (2)

Surds

Surds are expressions with irrational square roots in them (remember from p.2 that irrational numbers are ones which can't be written as fractions, such as most square roots, cube roots and π).

Manipulating Surds — 6 Rules to Learn **H**

There are 6 rules you need to learn for dealing with surds...

1. $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ e.g. $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$ — also $(\sqrt{b})^2 = \sqrt{b} \times \sqrt{b} = \sqrt{b \times b} = b$
2. $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ e.g. $\dfrac{\sqrt{8}}{\sqrt{2}} = \sqrt{\dfrac{8}{2}} = \sqrt{4} = 2$
3. $\sqrt{a} + \sqrt{b}$ — DO NOTHING — in other words it is definitely NOT $\sqrt{a+b}$
4. $(a + \sqrt{b})^2 = (a + \sqrt{b})(a + \sqrt{b}) = a^2 + 2a\sqrt{b} + b$ — NOT just $a^2 + (\sqrt{b})^2$ (see p.18)
5. $(a + \sqrt{b})(a - \sqrt{b}) = a^2 + a\sqrt{b} - a\sqrt{b} - (\sqrt{b})^2 = a^2 - b$ (see p.19).
6. $\dfrac{a}{\sqrt{b}} = \dfrac{a}{\sqrt{b}} \times \dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a\sqrt{b}}{b}$ This is known as 'RATIONALISING the denominator' — it's where you get rid of the $\sqrt{\ }$ on the bottom of the fraction. For denominators of the form $a \pm \sqrt{b}$, you always multiply by the denominator but change the sign in front of the root (see example 3 below).

Use the Rules to Simplify Expressions

EXAMPLES:

1. Write $\sqrt{300} + \sqrt{48} - 2\sqrt{75}$ in the form $a\sqrt{3}$, where a is an integer.

Write each surd in terms of $\sqrt{3}$:
$\sqrt{300} = \sqrt{100 \times 3} = \sqrt{100} \times \sqrt{3} = 10\sqrt{3}$
$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3}$
$2\sqrt{75} = 2\sqrt{25 \times 3} = 2 \times \sqrt{25} \times \sqrt{3} = 10\sqrt{3}$

Then do the sum (leaving your answer in terms of $\sqrt{3}$):
$\sqrt{300} + \sqrt{48} - 2\sqrt{75} = 10\sqrt{3} + 4\sqrt{3} - 10\sqrt{3} = 4\sqrt{3}$

2. A rectangle with length $4x$ cm and width x cm has an area of 32 cm². Find the exact value of x, giving your answer in its simplest form.

Area of rectangle = length × width = $4x \times x = 4x^2$
So $4x^2 = 32$
$x^2 = 8$
$x = \pm\sqrt{8}$ — You can ignore the negative square root (see p.22) as length must be positive.

'Exact value' means you have to leave your answer in surd form, so get $\sqrt{8}$ into its simplest form:
$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4}\sqrt{2}$
$= 2\sqrt{2}$ So $x = 2\sqrt{2}$

3. Write $\dfrac{3}{2+\sqrt{5}}$ in the form $a + b\sqrt{5}$, where a and b are integers.

To rationalise the denominator, multiply top and bottom by $2 - \sqrt{5}$:
$\dfrac{3}{2+\sqrt{5}} = \dfrac{3(2-\sqrt{5})}{(2+\sqrt{5})(2-\sqrt{5})}$
$= \dfrac{6 - 3\sqrt{5}}{2^2 - 2\sqrt{5} + 2\sqrt{5} - (\sqrt{5})^2}$
$= \dfrac{6-3\sqrt{5}}{4-5} = \dfrac{6-3\sqrt{5}}{-1} = -6 + 3\sqrt{5}$ (so a = −6 and b = 3)

THE BIG QUESTION

Show that $\dfrac{5+2\sqrt{3}}{2+\sqrt{3}}$ can be written as $4 - \sqrt{3}$

Sequences

What do I need to be able to do?

By the end of this unit you should be able to:
- Understand factors and multiples
- Express numbers as a product of primes
- Find the HCF and LCM
- Describe and continue sequences
- Explore sequences
- Find the nth term of a linear sequence

Keywords

Factor: numbers we multiply together to make another number
Multiple: the result of multiplying a number by an integer
HCF: highest common factor. The biggest factor that numbers share.
LCM: lowest common multiple. The first multiple numbers share.
Arithmetic: a sequence where the difference between the terms is constant
Geometric: a sequence where each term is found by multiplying the previous one by a fixed nonzero number
Sequence: items or numbers put in a pre-decided order

Multiples — The "times table" of a given number (R)

All the numbers in this lists below are multiples of 3.

3, 6, 9, 12, 15…

This list continues and doesn't end

$3x, 6x, 9x$ …

x could take any value and as the variable is a multiple of 3 the answer will also be a multiple of 3

Non example of a multiple

4.5 is not a multiple of 3 because it is 3 x 1.5 ← Not an integer

Factors (R)

Arrays can help represent factors

5 x 2 or 2 x 5 — **Factors of 10**: 1, 2, 5, 10 — 10 x 1 or 1 x 10

The number itself is always a factor

Factors and expressions

$6x \times 1$ OR $6 \times x$

$2x \times 3$

$3x \times 2$

Factors of $6x$: $6, x, 1, 6x, 2x, 3, 3x, 2$

Prime numbers

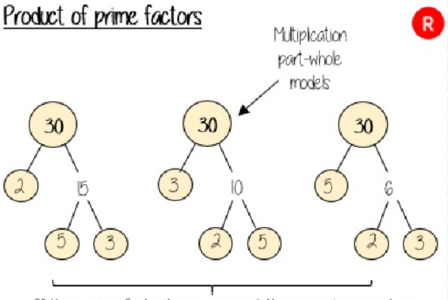
- Integer
- Only has 2 factors
- and itself

2 — The first prime number. The only even prime number

Learn or how-to quick recall…

2, 3, 5, 7, 11, 13, 17, 19, 23, 29… (R)

Product of prime factors (R)



All three prime factor trees represent the same decomposition

30 = 2 x 3 x 5 ← Multiplication of prime factors

Using prime factors for predictions

eg 60   30 x 2   2 x 3 x 5 x **2**
150   30 x 5   2 x 3 x 5 x **5**

Finding the HCF and LCM (R)

HCF – Highest common factor

HCF of 18 and 30

18: 1, 2, 3, (6), 9, 18
30: 1, 2, 3, 5, (6), 10, 15, 30

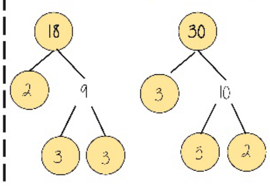
6 is the biggest factor they share — HCF = 6

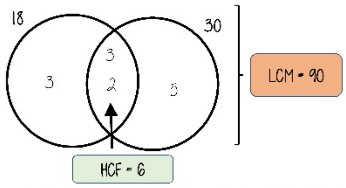
LCM – Lowest common multiple

LCM of 18 and 30

18: 18, 36, 54, 72, (90)
30: 30, 60, (90)

The first time their multiples match — LCM = 90





Arithmetic/ Geometric sequences

Arithmetic Sequences change by a common difference. This is found by addition or subtraction between terms

Geometric Sequences change by a common ratio. This is found my multiplication/ division between terms

Term to term rule – how you get from one term (number in the sequence) to the next term

Position to term rule – take the rule and substitute in a position to find a term. Eg. Multiply the position number by 3 and then add 2

Other sequences

Fibonacci Sequence — Each term is the sum of the previous two terms

1, 1, 2, 3, 5, 8 …

Triangular Numbers – look at the formation

1, 3, 6, 10, 15 …

Square Numbers – look at the formation

1, 4, 9, 16…

Sequences are the repetition of a pattern

Finding the nth term (R)

This is the 4 times table → 4, 8, 12, 16, 20….

4n

7, 11, 15, 19, 22

This has the same constant difference – but is 3 more than the original sequence

4n + 3

This is the constant difference between the terms in the sequence

This is the comparison (difference) between the original and new sequence

Finding the nth Term of a Quadratic Sequence **H**

A quadratic sequence has an n^2 term — the difference between the terms changes as you go through the sequence, but the difference between the differences is the same each time.

EXAMPLE: Find an expression for the nth term of the sequence that starts 10, 14, 20, 28...

n: 1 2 3 4

term: 10 14 20 28

+4 +6 +8

+2 +2 — So the expression will contain an n^2 term.

term:	10	14	20	28
n^2:	1	4	9	16
term − n^2:	9	10	11	12

The expression for this linear sequence is $n + 8$

So the expression for the nth term is $n^2 + n + 8$

1) Find the difference between each pair of terms.
2) The difference is changing, so work out the difference between the differences.
3) Divide this value by 2 — this gives the coefficient of the n^2 term (here it's $2 \div 2 = 1$).
4) Subtract the n^2 term from each term in the sequence. This will give you a linear sequence.
5) Find the rule for the nth term of the linear sequence (see above) and add this on to the n^2 term.

Again, make sure you check your expression by putting the first few values of n back in — so $n = 1$ gives $1^2 + 1 + 8 = 10$, $n = 2$ gives $2^2 + 2 + 8 = 14$ and so on.

THE BIG QUESTION

Here are the first 5 terms of a Fibonacci sequence.

2 2 4 6 10

Find the 8th term of this sequence.

Here are the first 5 terms of a sequence.

9 14 19 24 29

Find an expression, in terms of n, for the nth term of this sequence.

Here are the first 5 terms of a quadratic sequence.

19 15 9 1 -9

Find an expression, in terms of n, for the nth term of this sequence.

42

6 + 10 = 16
10 + 16 = 26
16 + 26 = 42

5n + 4

5n 5 10 15 20 25

$-n^2 - n + 21$

19 15 9 1 -9
-4 -6 -8 -10
-2 -2 -2

2a = -2, a = -1
3a + b = -4, 3(-1) + b = -4, -3 + b = -4, b = -1
a + b + c = 19, -1 - 1 + c = 19, -2 + c = 19, c = 21

Vectors

_maths

What do I need to be able to do?

By the end of this unit you should be able to:

- Understand and represent vectors
- Use and read vector notation
- Draw and understand vectors multiplied by a scalar
- Draw and understand addition of vectors
- Draw and understand addition and subtraction of vectors

Keywords

Direction: the line our course something is going
Magnitude: the magnitude of a vector is its length
Scalar: a single number used to represent the multiplier when working with vectors
Column vector: a matrix of one column describing the movement from a point
Resultant: the vector that is the sum of two or more other vectors
Parallel: straight lines that never meet

Understand and represent vectors

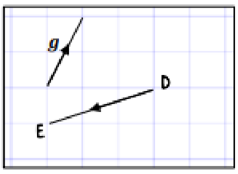
Vectors show both direction and magnitude

Column vectors have been seen in translations to describe the movement of one image onto another

Movement along the x-axis → $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ ← Movement along the y-axis

The arrow is pointing in the direction from starting point to end point of the vector	The direction is important to correctly write the vector
The magnitude is the length of the vector (This is calculated using Pythagoras theorem and forming a right-angled triangle with auxiliary lines)	The magnitude stays the same even if the direction changes

Understand and represent vectors



Vector notation $\overrightarrow{DE}$ is another way to represent the vector joining the point D to the point E.

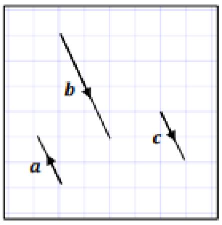
$\overrightarrow{DE} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

The arrow also indicates the direction from point D to point E.

Vectors can also be written in bold lower case so $\boldsymbol{g}$ represents the vector $\boldsymbol{g} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Vectors multiplied by a scalar

Parallel vectors are scalar multiples of each other



$\boldsymbol{b} = 2 \times \boldsymbol{c} = 2\boldsymbol{c}$

Multiply $\boldsymbol{c}$ by 2 this becomes $\boldsymbol{b}$. The two lines are parallel

$\boldsymbol{a} = -1 \times \boldsymbol{c} = -\boldsymbol{c}$

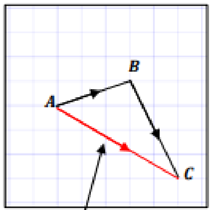
The vectors $\boldsymbol{a}$ and $\boldsymbol{c}$ are also parallel. A negative scalar causes the vector to reverse direction

$\boldsymbol{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\boldsymbol{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ $\boldsymbol{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\boldsymbol{b} = -2 \times \boldsymbol{a} = -2\boldsymbol{a}$

Addition of vectors

$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$



The resultant

$\overrightarrow{AB} + \overrightarrow{BC}$

$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

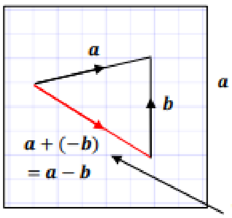
$= \begin{pmatrix} 3+2 \\ 1+-4 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

Look how this addition compares to the vector $\overrightarrow{AC}$

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

Addition and subtraction of vectors



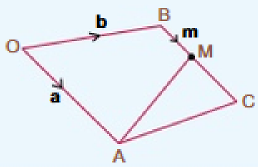
$\boldsymbol{a} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ $\boldsymbol{b} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$\boldsymbol{a} + (-\boldsymbol{b}) = \begin{pmatrix} 5 + -0 \\ 1 + -4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

The resultant is $\boldsymbol{a} - \boldsymbol{b}$ because the vector is in the opposite direction to b which needs a scalar of -1

EXAMPLE:

In the diagram below, M is the midpoint of BC.
Find vectors $\overrightarrow{AM}$, $\overrightarrow{OC}$ and $\overrightarrow{AC}$ in terms of a, b and m.



To obtain the unknown vector just 'get there' by any route made up of known vectors.

$\overrightarrow{AM} = -\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{m}$ — **A to M via O and B**

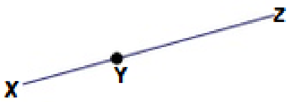
$\overrightarrow{OC} = \underset{\sim}{b} + 2\underset{\sim}{m}$ — **O to C via B and M — M's half-way between B and C so $\overrightarrow{BC} = 2\underset{\sim}{m}$**

$\overrightarrow{AC} = -\underset{\sim}{a} + \underset{\sim}{b} + 2\underset{\sim}{m}$ — **A to C via O, B and M**

Extra bits and pieces can crop up in vector questions — these examples will show you how to tackle them...

Vectors Along a Straight Line **H**

1) You can use vectors to show that points lie on a straight line.
2) You need to show that the vectors along each part of the line point in the same direction — i.e. they're scalar multiples of each other.

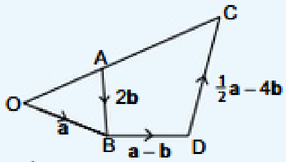


If XYZ is a straight line then $\overrightarrow{XY}$ must be a scalar multiple of $\overrightarrow{YZ}$.

EXAMPLE:

In the diagram, $\overrightarrow{OB} = \mathbf{a}$, $\overrightarrow{AB} = 2\mathbf{b}$, $\overrightarrow{BD} = \mathbf{a} - \mathbf{b}$ and $\overrightarrow{DC} = \frac{1}{2}\mathbf{a} - 4\mathbf{b}$.

Show that OAC is a straight line.



1) Work out the vectors along the two parts of OAC (OA and AC) using the vectors you know.

$\overrightarrow{OA} = \underset{\sim}{a} - 2\underset{\sim}{b}$

$\overrightarrow{AC} = 2\underset{\sim}{b} + (\underset{\sim}{a} - \underset{\sim}{b}) + \left(\frac{1}{2}\underset{\sim}{a} - 4\underset{\sim}{b}\right)$

$= \frac{3}{2}\underset{\sim}{a} - 3\underset{\sim}{b} = \frac{3}{2}(\underset{\sim}{a} - 2\underset{\sim}{b})$

2) Check that $\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{OA}$.

So, $\overrightarrow{AC} = \frac{3}{2}\overrightarrow{OA}$.

3) Explain why this means OAC is a straight line.

$\overrightarrow{AC}$ is a scalar multiple of $\overrightarrow{OA}$, so OAC must be a straight line.

Vector Questions Can Involve Ratios

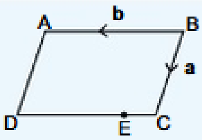
Ratios are used in vector questions to tell you the lengths of different sections of a straight line. If you know the vector along part of that line, you can use this information to find other vectors along the line.

E.g.


XY : YZ = 2 : 3 tells you that $\overrightarrow{XY} = \frac{2}{5}\overrightarrow{XZ}$ and $\overrightarrow{YZ} = \frac{3}{5}\overrightarrow{XZ}$.

EXAMPLE:

ABCD is a parallelogram, with AB parallel to DC and AD parallel to BC.
Point E lies on DC, such that DE : EC = 3 : 1.
$\overrightarrow{BC} = \mathbf{a}$ and $\overrightarrow{BA} = \mathbf{b}$.
Find $\overrightarrow{AE}$ in terms of **a** and **b**.



1) Write $\overrightarrow{AE}$ as a route along the parallelogram.

$\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE}$

2) Use the parallel sides to find $\overrightarrow{AD}$ and $\overrightarrow{DC}$.

$\overrightarrow{AD} = \overrightarrow{BC} = \underset{\sim}{a}$

$\overrightarrow{DC} = \overrightarrow{AB} = -\underset{\sim}{b}$

3) Use the ratio to find $\overrightarrow{DE}$.

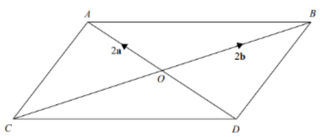
$\overrightarrow{DE} = \frac{3}{4}\overrightarrow{DC} = \frac{3}{4}(-\underset{\sim}{b}) = -\frac{3}{4}\underset{\sim}{b}$

4) Now use $\overrightarrow{AD}$ and $\overrightarrow{DE}$ to find $\overrightarrow{AE}$.

So $\overrightarrow{AE} = \overrightarrow{AD} + \overrightarrow{DE} = \underset{\sim}{a} - \frac{3}{4}\underset{\sim}{b}$

THE BIG QUESTION

The diagram shows a parallelogram.



$\overrightarrow{OA} = 2\mathbf{a}$

$\overrightarrow{OB} = 2\mathbf{b}$

(a) Find, in terms of **a**, the vector $\overrightarrow{DA}$

.... (1)

(b) Find, in terms of **a** and **b**, the vector $\overrightarrow{AB}$

.... (1)

(c) Find, in terms of **a** and **b**, the vector $\overrightarrow{AC}$

Loci and constructions

What do I need to be able to do?

By the end of this unit you should be able to:

- Draw and measure angles
- Construct scale drawings
- Find locus of distance from points, lines, two lines
- Construct perpendiculars from points, lines, angles
- Identify congruence
- Identify congruent triangles

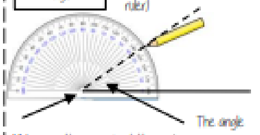
Keywords

Protractor: piece of equipment used to measure and draw angles
Locus: set of points with a common property
Equidistant: the same distance
Discorectangle: (a stadium) – a rectangle with semi circles at either end
Perpendicular: lines that meet at 90°
Arc: part of a curve
Bisector: a line that divides something into two equal parts
Congruent: the same shape and size

Draw and measure angles (R)

Draw a 35° angle

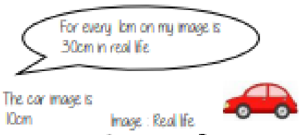
Make a mark at 35° with a pencil. And join to the angle point (use a ruler)



Make sure the cross is at the end of the line (where you want the angle)

Scale drawings (R)

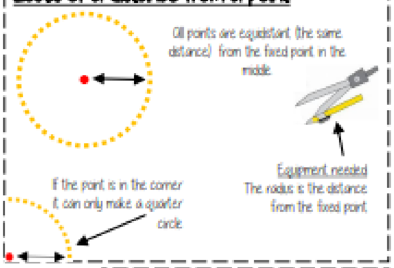
A picture of a car is drawn with a scale of 1:30



The car image is 10cm

Image : Real life
1cm : 30cm
10cm : 300cm
(×10 on both sides)

Locus of a distance from a point

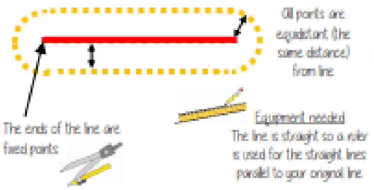


All points are equidistant (the same distance) from the fixed point in the middle

If the point is in the corner it can only make a quarter circle

Equipment needed
The radius is the distance from the fixed point

Locus of a distance from a straight line



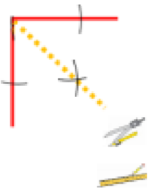
All points are equidistant (the same distance) from line

The ends of the line are fixed points

Equipment needed
The line is straight so a ruler is used for the straight lines parallel to your original line

Locus of a distance from two lines

Also an angle bisector
This cuts the angle in half



From the angle vertex draw two arcs that cut the lines forming the angle

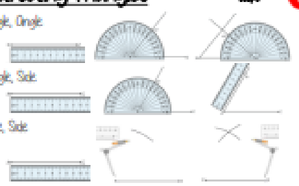
Keep the compass the same size and use the new arcs as centres to draw intersecting arcs in the middle

Join the vertex to the intersection

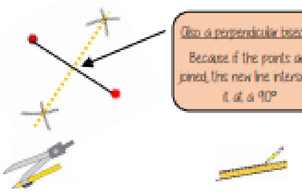
Constructing Triangles (R)

Link to shape

Side, Angle, Angle
Side, Angle, Side
Side, Side, Side



Locus equidistant from two points

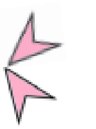


Also a perpendicular bisector
Because if the points are joined, this new line intersects it at a 90°

Keep the compass the same size and draw two arcs from each point

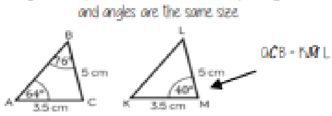
Join the intersections with a ruler
All points on this line are equidistant from both points

Congruent figures



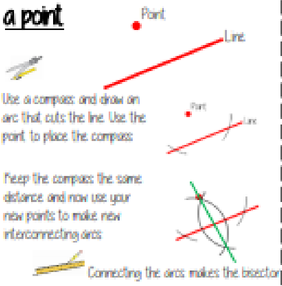
Congruent figures are identical in size and shape – they can be reflections or rotations of each other

Congruent shapes are identical – all corresponding sides and angles are the same size



Because all the angles are the same and AC=KM BC=LM triangles ABC and KLM are **congruent**

Construct a perpendicular from a point



Use a compass and draw an arc that cuts the line. Use the point to place the compass

Keep the compass the same distance and now use your new points to make new interconnecting arcs

Connecting the arcs makes the bisector

If P is a point on the line the steps are the same

Congruent triangles

Side-side-side
All three sides on the triangle are the same size

Angle-side-angle
Two angles and the side connecting them are equal in two triangles

Side-angle-side
Two sides and the angle in-between them are equal in two triangles (it will also mean the third side is the same size on both shapes)

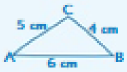
Right angle-hypotenuse-side
The triangles both have a right angle, the hypotenuse and one side are the same

How you construct a triangle depends on what info you're given about the triangle...

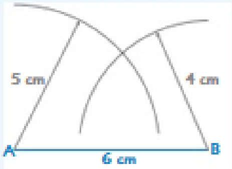
Three sides — use a *Ruler and Compasses*

EXAMPLE: Construct the triangle ABC where AB = 6 cm, BC = 4 cm, AC = 5 cm.

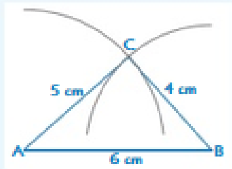
First, sketch and label a triangle so you know roughly what's needed. It doesn't matter which line you make the base line.



Draw the base line accurately. Label the ends A and B.



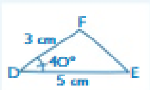
For AC, set the compasses to 5 cm, put the point at A and draw an arc. For BC, set the compasses to 4 cm, put the point at B and draw an arc.



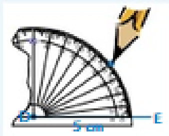
Where the arcs cross is point C. Now you can finish your triangle.

Sides and *Angles* — use a *Ruler and Protractor*

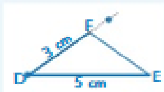
EXAMPLE: Construct triangle DEF. DE = 5 cm, DF = 3 cm, and angle EDF = 40°.



Roughly sketch and label the triangle.

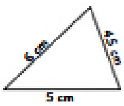


Draw the base line accurately. Then draw angle EDF (the angle at D) — place the centre of the protractor over D, measure 40° and put a dot.

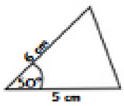


Measure 3 cm towards the dot and label it F. Join up D and F. Now you've drawn the two sides and the angle. Just join up F and E to complete the triangle.

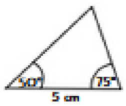
If you're given 3 pieces of information about a triangle, there's usually only one triangle that you could draw.



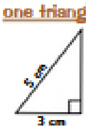
SSS — 3 sides



SAS — 2 sides and the angle between them.



ASA — 2 angles and the side between them.



RHS — right angle, the hypotenuse and another side.

However, if you're given 2 sides and an angle which isn't between them, there are TWO possible triangles you could draw.

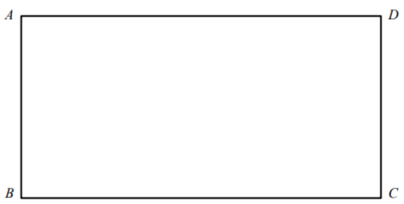
The 5 cm side could be in either of the positions shown.

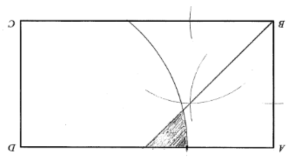
THE BIG QUESTION

Here is a scale drawing of a room.
The scale is 1 cm to 2 m.

A chair is going to be placed in the room.
The chair must be closer to *AB* than *BC*.
The chair must be less than 14 m from *D*.

Shade the region where the chair can be placed.





Inequalities and Graphs

What do I need to be able to do?

By the end of this unit you should be able to:

- Draw quadratic graphs
- Interpret quadratic graphs
- Interpret other graphs including reciprocals
- Represent inequalities

Keywords

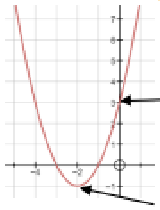
Quadratic: a curved graph with the highest power being 2 Square power
Inequality: makes a non equal comparison between two numbers
Reciprocal: a reciprocal is 1 divided by the number
Cubic: a curved graph with the highest power being 3 Cubic power
Origin: the coordinate (0, 0)
Parabola: a 'u' shaped curve that has mirror symmetry

Quadratic Graphs

$y = x^2 + 4x + 3$

If x^2 is the highest power in your equation then you have a quadratic graph

It will have a parabola shape



Intersection with the y axis

Substitute the x values into the equation of your line to find the y coordinates

x	-4	-3	-2	-1	0	1
y	3	0	-1	0	3	8

Coordinate pairs for plotting $(-3, 0)$

Plot all of the coordinate pairs and join the points with a curve (freehand)

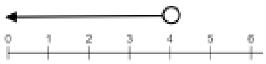
Quadratic graphs are always symmetrical with the turning point in the middle

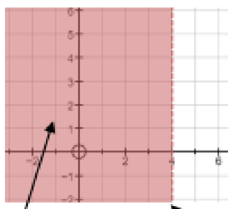
Represent Inequalities

Multiple methods of representing inequalities

$x < 4$

All values are less than 4



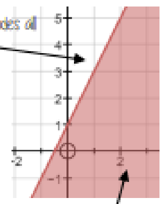


The shaded area indicates all possible values of x

The dotted line shows that the inequality does not include these points

The solid line shows that the inequality includes all the points on this line

$y \geq 2x + 1$

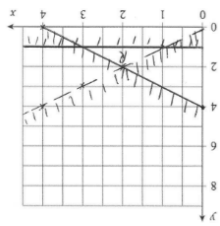


The shaded area indicates all possible solutions to this inequality

THE BIG QUESTION

On the grid, clearly indicate the region that satisfies all these inequalities.

$y < x$ $\quad$ $y \geq 1$ $\quad$ $x + y \leq 4$



What do I need to be able to do?

By the end of this unit you should be able to:

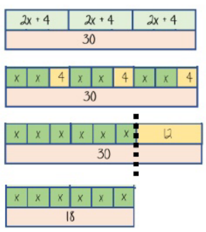
- Form and solve equations and inequalities
- Represent and interpret solutions on a number line as inequalities
- Draw straight line graphs and find solutions to equations
- Form and solve equations and inequalities with unknowns on both sides

Keywords

Solution: a value we can put in place of a variable that makes the equation true
Variable: a symbol for a number we don't know yet
Equation: an equation says that two things are equal – it will have an equals sign =
Expression: numbers, symbols and operators grouped together to show the value of something
Identity: An equation where both sides have variables that cause the same answer includes ≡
Linear: an equation or function that is the equation of a straight line
Intersection: the point that two lines meet
Inequality: an inequality compares two values showing if one is greater than, less than or equal to another

Solve equations (R)

$3(2x + 4) = 30$



$3(2x + 4) = 30$ — Expand the brackets

$6x + 12 = 30$
$-12 \quad -12$

$6x = 18$
$\div 6 \quad \div 6$

$x = 3$

Substitute to check your answer. This could be negative or a fraction or decimal

Form and solve inequalities (R)

"Two more than treble my number is greater than 11"

Form: x → ×3 → +2 → 11

$3x + 2 > 11$

Solve: x ← -3 ← -2 ← 11

$x > 3$

Solutions on a number line

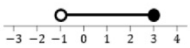
$x < 1$ $x \leq 1$ $x > 1$ $x \geq 1$

Both represent values less than 1. Includes the value 1.

Both represent values more than 1. Includes the value 1.

● Includes the value it sits above
○ Does NOT include the value it sits above

"Values less than or equal to 3 but also more than -1"



$-1 < x \leq 3$

This includes the integer values 0,1,2,3

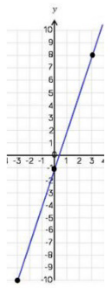
Plotting straight line graphs (R)

$y = 3x - 1$ → 3 × the x coordinate then – 1

x	-3	0	3
y	-10	-1	8

Draw a table to display this information

This represents a coordinate pair (-3, -10)



You only need two points to form a straight line

Plotting more points helps you decide if your calculations are correct (if they do make a straight line)

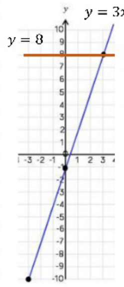
Remember to join the points to make a line

Find solutions graphically

For linear equations there is only one point the graph meets the x value

$x = 2$
$y = 4$

These two lines will cross at (2,4) because they are just x= and y= they are parallel to axes and meet in one place



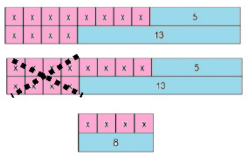
$3x - 1 = 8$

Remember equation of a line format is $y = mx + c$

The solution is the point the two lines meet (3,8)

Equations: unknown on both sides (R)

$8x + 5 = 4x + 13$



$8x + 5 = 4x + 13$
$-4x \quad -4x$
$4x + 5 = 13$
$-5 \quad -5$
$4x = 8$
$\div 4 \quad \div 4$
$x = 2$

Inequalities: unknown on both sides

$8x + 5 \leq 4x + 13$ → $x \leq 2$

Only value 2 or less will satisfy this inequality

THE BIG QUESTION

Solve $4(2x + 1) > 9$

Brackets and Factorising

I usually use brackets to make witty comments (I'm very witty), but in algebra they're useful for simplifying things. First of all, you need to know how to expand brackets (multiply them out).

Single Brackets

The main thing to remember when multiplying out brackets is that the thing outside the bracket multiplies each separate term inside the bracket.

EXAMPLE: Expand the following:

a) $4a(3b - 2c)$
$= (4a \times 3b) + (4a \times -2c)$
$= 12ab - 8ac$

b) $-4(3p^2 - 7q^3)$
$= (-4 \times 3p^2) + (-4 \times -7q^3)$
$= -12p^2 + 28q^3$

Note: both signs have been reversed.

Double Brackets

Double brackets are trickier than single brackets — this time, you have to multiply everything in the first bracket by everything in the second bracket. You'll get 4 terms, and usually 2 of them will combine to leave 3 terms. There's a handy way to multiply out double brackets — it's called the FOIL method:

First — multiply the first term in each bracket together
Outside — multiply the outside terms (i.e. the first term in the first bracket by the second term in the second bracket)
Inside — multiply the inside terms (i.e. the second term in the first bracket by the first term in the second bracket)
Last — multiply the second term in each bracket together

EXAMPLE: Expand and simplify $(2p - 4)(3p + 1)$

$(2p - 4)(3p + 1) = (2p \times 3p) + (2p \times 1) + (-4 \times 3p) + (-4 \times 1)$
$= 6p^2 + 2p - 12p - 4$
$= 6p^2 - 10p - 4$

The two p terms combine together.

Always write out SQUARED BRACKETS as TWO BRACKETS (to avoid mistakes), then multiply out as above.
So $(3x + 5)^2 = (3x + 5)(3x + 5) = 9x^2 + 15x + 15x + 25 = 9x^2 + 30x + 25$.
(DON'T make the mistake of thinking that $(3x + 5)^2 = 9x^2 + 25$ — this is wrong wrong wrong.)

Right, now you know how to expand brackets, it's time to put them back in. This is known as factorising.

Factorising — Putting Brackets In

This is the exact reverse of multiplying out brackets. Here's the method to follow:

1) Take out the biggest number that goes into all the terms.
2) For each letter in turn, take out the highest power (e.g. x, x^2 etc.) that will go into EVERY term.
3) Open the bracket and fill in all the bits needed to reproduce each term.
4) Check your answer by multiplying out the bracket and making sure it matches the original expression.

EXAMPLES:

1. Factorise $3x^2 + 6x$

Biggest number that'll divide into 3 and 6; Highest power of x that will go into both terms

$3x(x + 2)$

Check: $3x(x + 2) = 3x^2 + 6x$ ✓

2. Factorise $8x^3y + 2xy^2$

Biggest number that'll divide into 8 and 2; Highest powers of x and y that will go into both terms

$2xy(4x^2 + y)$

Check: $2xy(4x^2 + y) = 8x^3y + 2xy^2$ ✓

REMEMBER: The bits taken out and put at the front are the common factors. The bits inside the bracket are what's needed to get back to the original terms if you multiply the bracket out again.

THE BIG QUESTION

(a) Factorise fully $18a^2bc + 30abc^2$

(b) Expand and Simplify $4(2y - 7) - 3(5y - 3)$

$-7y - 19$ (2)

$6abc(3a + 5c)$ (2)

D.O.T.S. — The Difference Of Two Squares

The 'difference of two squares' (D.O.T.S. for short) is where you have 'one thing squared' take away 'another thing squared'. There's a quick and easy way to factorise it — just use the rule below:

$$a^2 - b^2 = (a + b)(a - b)$$

EXAMPLE: Factorise:

a) $9p^2 - 16q^2$ Answer: $9p^2 - 16q^2 = (3p + 4q)(3p - 4q)$
Here you had to spot that 9 and 16 are square numbers.

b) $3x^2 - 75y^2$ Answer: $3x^2 - 75y^2 = 3(x^2 - 25y^2) = 3(x + 5y)(x - 5y)$
This time, you had to take out a factor of 3 first.

c) $x^2 - 5$ Answer: $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$
Although 5 isn't a square number, you can write it as $(\sqrt{5})^2$.

EXAMPLE: Simplify $\frac{x^2 - 36}{5x + 30}$

$$\frac{x^2 - 36}{5x + 30} = \frac{(x+6)(x-6)}{5(x+6)} = \frac{x-6}{5}$$

The numerator is a difference of two squares.
Factorise the denominator.

There are several ways of solving a quadratic equation as detailed on the following pages.

Factorising a Quadratic

1) 'Factorising a quadratic' means 'putting it into 2 brackets'.
2) The standard format for quadratic equations is: $ax^2 + bx + c = 0$.
3) If $a = 1$, the quadratic is much easier to deal with. E.g. $x^2 + 3x + 2 = 0$
4) As well as factorising a quadratic, you might be asked to solve the equation. This just means finding the values of x that make each bracket 0 (see example below).

See next page for when 'a' is not 1.

Factorising Method when a = 1

1) ALWAYS rearrange into the STANDARD FORMAT: $x^2 + bx + c = 0$.
2) Write down the TWO BRACKETS with the x's in: $(x\quad)(x\quad) = 0$.
3) Then find 2 numbers that MULTIPLY to give 'c' (the end number) but also ADD/SUBTRACT to give 'b' (the coefficient of x). — Ignore any minus signs at this stage.
4) Fill in the +/– signs and make sure they work out properly.
5) As an ESSENTIAL CHECK, expand the brackets to make sure they give the original equation.
6) Finally, SOLVE THE EQUATION by setting each bracket equal to 0.

You only need to do step 6) if the question asks you to solve the equation — if it just tells you to factorise, you can stop at step 5).

EXAMPLE: Solve $x^2 - x = 12$.

1) $x^2 - x - 12 = 0$ ← 1) Rearrange into the standard format.

2) $(x\quad)(x\quad) = 0$ ← 2) Write down the initial brackets.

3) ← 3) Find the right pairs of numbers that multiply to give c (= 12), and add or subtract to give b (= 1) (remember, we're ignoring the +/– signs for now).

1 × 12	Add/subtract to give:	13 or 11
2 × 6	Add/subtract to give:	8 or 4
3 × 4	Add/subtract to give:	7 or 1

$(x\quad 3)(x\quad 4) = 0$ This is what we want.

4) $(x + 3)(x - 4) = 0$ ← 4) Now fill in the +/– signs so that 3 and 4 add/subtract to give –1 (= b).

5) Check: $(x + 3)(x - 4) = x^2 - 4x + 3x - 12 = x^2 - x - 12$ ✓ ← 5) ESSENTIAL check — EXPAND the brackets to make sure they give the original expression.

But we're not finished yet — we've only factorised it, we still need to...

6) $(x + 3) = 0 \Rightarrow x = -3$, $(x - 4) = 0 \Rightarrow x = 4$ ← 6) SOLVE THE EQUATION by setting each bracket equal to 0.

THE BIG QUESTION

(a) Expand and simplify $(3x - 5)(2x - 3)$

(b) Factorise $n^2 - 3n - 18$

$(n + 3)(n - 6)$

(2)
$6x^2 - 19x + 15$

So far so good. It gets a bit more complicated when 'a' isn't 1, but it's all good fun, right? Right? Well, I think it's fun anyway.

When 'a' is Not 1 — H

The basic method is still the same but it's a bit messier — the initial brackets are different as the first terms in each bracket have to multiply to give 'a'. This means finding the other numbers to go in the brackets is harder as there are more combinations to try. The best way to get to grips with it is to have a look at an example.

EXAMPLE: Solve $3x^2 + 7x - 6 = 0$.

1) $3x^2 + 7x - 6 = 0$
2) $(3x \quad)(x \quad) = 0$
3) Number pairs: 1×6 and 2×3

(3x 1)(x 6) multiplies to give 18x and 1x which add/subtract to give 17x or 19x
(3x 6)(x 1) multiplies to give 3x and 6x which add/subtract to give 9x or 3x
(3x 3)(x 2) multiplies to give 6x and 3x which add/subtract to give 9x or 3x
(3x 2)(x 3) multiplies to give 9x and 2x which add/subtract to give 11x or 7x ✓

$(3x \quad 2)(x \quad 3)$

4) $(3x - 2)(x + 3)$
5) $(3x - 2)(x + 3) = 3x^2 + 9x - 2x - 6 = 3x^2 + 7x - 6$ ✓
6) $(3x - 2) = 0 \Rightarrow x = \frac{2}{3}$
$(x + 3) = 0 \Rightarrow x = -3$

1) Rearrange into the standard format.
2) Write down the initial brackets — this time, one of the brackets will have a 3x in it.
3) The tricky part: first, find pairs of numbers that multiply to give c (= 6), ignoring the minus sign for now.
Then, try out the number pairs you just found in the brackets until you find one that gives 7x. But remember, each pair of numbers has to be tried in 2 positions (as the brackets are different — one has 3x in it).
4) Now fill in the +/– signs so that 9 and 2 add/subtract to give +7 (= b).
5) ESSENTIAL check — EXPAND the brackets.
6) SOLVE THE EQUATION by setting each bracket equal to 0 (if a isn't 1, one of your answers will be a fraction).

EXAMPLE: Solve $2x^2 - 9x = 5$.

1) Put in standard form: $2x^2 - 9x - 5 = 0$
2) Initial brackets: $(2x \quad)(x \quad) = 0$
3) Number pairs: 1×5

(2x 5)(x 1) multiplies to give 2x and 5x which add/subtract to give 3x or 7x
(2x 1)(x 5) multiplies to give 1x and 10x which add/subtract to give 9x or 11x ✓

$(2x \quad 1)(x \quad 5)$

4) Put in the signs: $(2x + 1)(x - 5)$
5) Check: $(2x + 1)(x - 5) = 2x^2 - 10x + x - 5 = 2x^2 - 9x - 5$ ✓
6) Solve: $(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}$
$(x - 5) = 0 \Rightarrow x = 5$

THE BIG QUESTION

Solve $5x^2 + 11x - 12 = 0$

Factorise $3x^2 + 16x + 21$

$(x + 3)(5x - 4)$

$x = -3 \quad x = \frac{4}{5}$

Transformations

What do I need to be able to do?

By the end of this unit you should be able to:

- Enlarge by a positive scale factor
- Enlarge by a fractional scale factor
- Identify similar shapes
- Work out missing sides and angles in similar shapes
- Use parallel lines to find missing angles
- Understand similarity and congruence

Keywords

Enlarge: to make a shape bigger (or smaller) by a given multiplier (scale factor)
Scale Factor: the multiplier of enlargement
Centre of enlargement: the point the shape is enlarged from
Similar: when one shape can become another with a reflection, rotation, enlargement or translation.
Congruent: the same size and shape
Corresponding: items that appear in the same place in two similar situations
Parallel: straight lines that never meet (equal gradients)

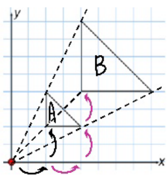
Positive scale factors (R)

Enlargement from a point

Enlarge shape A by SF 2 from (0,0)

The shape is enlarged by 2

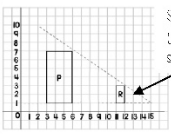
The distance from the point enlarges by 2



Fractional scale factors (R)

Fractions less than 1 make a shape SMALLER

R is an enlargement of P by a scale factor $\frac{1}{3}$ from centre of enlargement (15,1)

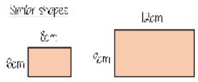


SF $\frac{1}{3}$ - R is three times smaller than P

Identify similar shapes

Angles in similar shapes do not change
e.g. if a triangle gets bigger the angles can not go above 180°

Similar shapes

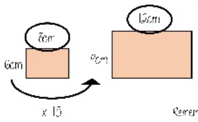


Scale factor:
Both sides on the bigger shape are 1.5 times bigger

Compare sides: 6 : 9 = 2 : 3 8 : 12 = 2 : 3

Both sets of sides are in the same ratio

Information in similar shapes



Compare the equivalent sides on both shapes

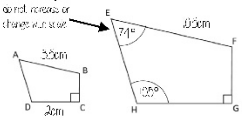
Scale factor is the multiplicative relationship between the two lengths

Shape ABCD and EFGH are similar

Notation helps us find the corresponding sides

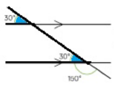
AB and EF are corresponding

Remember angles do not increase or change with scale



Angles in parallel lines (R)

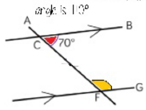
Alternate angles



Because alternate angles are equal the highlighted angles are the same size

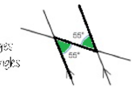
Co-interior angles

Because co-interior angles have a sum of 180° the highlighted angle is 110°



Corresponding angles

Because corresponding angles are equal the highlighted angles are the same size



As angles on a line add up to 180° co-interior angles can also be calculated from applying alternate / corresponding rules first

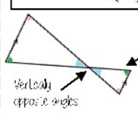
Similar triangles

Shares a vertex

Because corresponding angles are equal the highlighted angles are the same size

Parallel lines – all angles will be the same in both triangles

As all angles are the same this is similar – only one pair of sides are needed to show equality

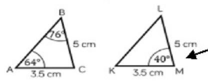


All the angles in both triangles are the same and so similar

Vertically opposite angles

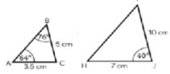
Congruence and Similarity

Congruent shapes are identical – all corresponding sides and angles are the same size



ACB = KML

Because all the angles are the same and AC=KM BC=LM triangles ABC and KLM are congruent



Because all angles are the same, but all sides are enlarged by 2 ABC and HIJ are similar

Conditions for congruent triangles

Triangles are congruent if they satisfy any of the following conditions

Side-side-side

All three sides on the triangle are the same size

Angle-side-angle

Two angles and the side connecting them are equal in two triangles

Side-angle-side

Two sides and the angle in-between them are equal in two triangles (it will also mean the third side is the same size on both shapes)

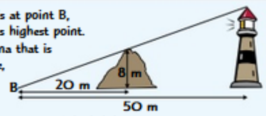
Right angle-hypotenuse-side

The triangles both have a right angle, the hypotenuse and one side are the same

Use Similarity to Find Missing Lengths (Grade 6)

You might have to use the properties of similar shapes to find missing distances, lengths etc. — you'll need to use scale factors (see p.81) to find the lengths of missing sides.

EXAMPLE: Suzanna is swimming in the sea. When she is at point B, she is 20 m from a rock that is 8 m tall at its highest point. There is a lighthouse 50 m away from Suzanna that is directly behind the rock. From her perspective, the top of the lighthouse is in line with the top of the rock. How tall is the lighthouse?



The triangles formed between Suzanna and the rock and Suzanna and the lighthouse are similar, so work out the scale factor: scale factor $= \frac{50}{20} = 2.5$

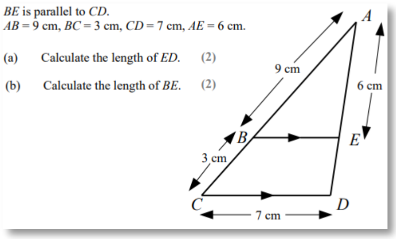
Now use the scale factor to work out the height of the lighthouse: height $= 8 \times 2.5 = 20$ m

THE BIG QUESTION

BE is parallel to *CD*.
AB = 9 cm, *BC* = 3 cm, *CD* = 7 cm, *AE* = 6 cm.

(a) Calculate the length of *ED*. (2)

(b) Calculate the length of *BE*. (2)



$7 \div \frac{4}{3}$

$7 \times \frac{3}{4} = \frac{21}{4}$ or 5.25

$\frac{21}{4}$

(Total for Question 1)

2 cm

There are four transformations you need to know — translation, rotation, reflection and enlargement.

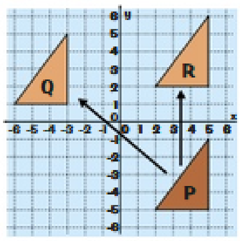
1) Translations

In a translation, the amount the shape moves by is given as a vector (see p.103-104) written $\begin{pmatrix}x\\y\end{pmatrix}$ — where x is the horizontal movement (i.e. to the right) and y is the vertical movement (i.e. up).
If the shape moves left and down, x and y will be negative.

EXAMPLE:
a) Describe the transformation that maps triangle P onto Q.
b) Describe the transformation that maps triangle P onto R.

a) To get from P to Q, you need to move 8 units left and 6 units up, so...
The transformation from P to Q is a translation by the vector $\begin{pmatrix}-8\\6\end{pmatrix}$.

b) The transformation from P to R is a translation by the vector $\begin{pmatrix}0\\7\end{pmatrix}$.



2) Rotations

To describe a rotation, you must give 3 details:
1) The angle of rotation (usually 90° or 180°).
2) The direction of rotation (clockwise or anticlockwise).
3) The centre of rotation (often, but not always, the origin).

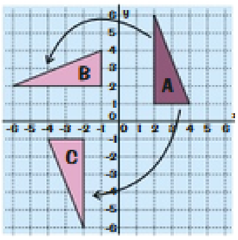
For a rotation of 180°, it doesn't matter whether you go clockwise or anticlockwise.

EXAMPLE:
a) Describe the transformation that maps triangle A onto B.
b) Describe the transformation that maps triangle A onto C.

a) The transformation from A to B is a rotation of 90° anticlockwise about the origin.

b) The transformation from A to C is a rotation of 180° clockwise (or anticlockwise) about the origin.

If it helps, you can use tracing paper to help you find the centre of rotation.

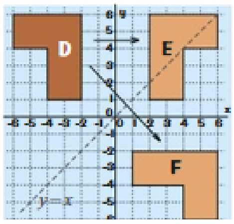


3) Reflections

For a reflection, you must give the equation of the mirror line.

EXAMPLE:
a) Describe the transformation that maps shape D onto shape E.
b) Describe the transformation that maps shape D onto shape F.

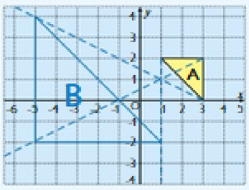
a) The transformation from D to E is a reflection in the y-axis.
b) The transformation from D to F is a reflection in the line $y = x$.



Scale Factors — Four Key Facts

1) If the scale factor is bigger than 1 the shape gets bigger.
2) If the scale factor is smaller than 1 (e.g. ½) it gets smaller.
3) If the scale factor is negative then the shape pops out the other side of the enlargement centre. If the scale factor is –1, it's exactly the same as a rotation of 180°.
4) The scale factor also tells you the relative distance of old points and new points from the centre of enlargement — this is very useful for drawing an enlargement, because you can use it to trace out the positions of the new points.

EXAMPLE: Enlarge shape A below by a scale factor of -3, centre (1, 1). Label the transformed shape B.

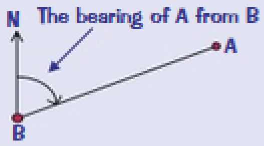


1) First, draw lines going through (1, 1) from each vertex of shape A.
2) Then, multiply the distance from each vertex to the centre of enlargement by 3, and measure this distance coming out the other side of the centre of enlargement.
So on shape A, vertex (3, 2) is 2 right and 1 up from (1, 1) — so the corresponding point on shape B will be 6 left and 3 down from (1, 1). Do this for every point.
3) Join the points you've drawn to form shape B.

Bearings

Bearings. They'll be useful next time you're off sailing. Or in your Maths exam.

Bearings



1) A bearing is just a direction given as an angle in degrees.
2) All bearings are measured clockwise from the North line.
3) All bearings are given as 3 figures:
   e.g. 060° rather than just 60°, 020° rather than 20° etc.

The 3 Key Words

To find or draw a bearing you must remember three key words:

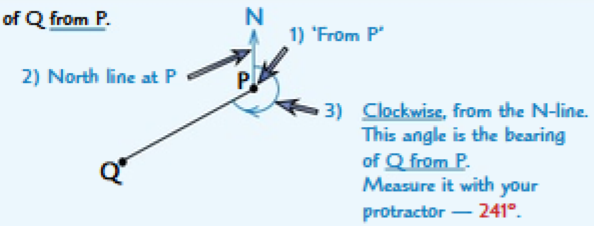
① 'FROM' — Find the word 'FROM' in the question, and put your pencil on the diagram at the point you are going 'from'.

② NORTH LINE — At the point you are going FROM, draw in a NORTH LINE.

③ CLOCKWISE — Now draw in the angle CLOCKWISE from the NORTH LINE to the line joining the two points — this angle is the bearing.

EXAMPLES:

1. Find the bearing of Q from P.

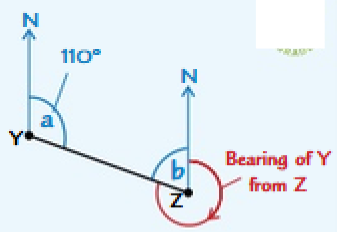


2. The bearing of Z from Y is 110°.
   Find the bearing of Y from Z.

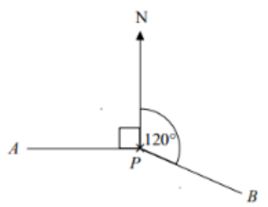
See page 89 for allied angles.

First sketch a diagram so you can see what's going on.
Angles a and b are allied, so they add up to 180°.

Angle b = 180° – 110° = 70°
So bearing of Y from Z = 360° – 70° = 290°.



THE BIG QUESTION



(a) Write down the bearing of B from P.

(b) Work out the bearing of A from P.

270

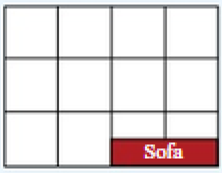
120

Scale drawing

Scale Drawings

Scale drawings work just like maps. To convert between real life and scale drawings, just replace the word 'map' with 'drawing' in the rules on the previous page.

EXAMPLE: This is a scale drawing of a room in Clare's house. 1 cm represents 1.5 m.



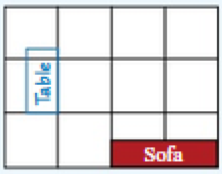
Scale drawings will often be shown on a grid.

a) Find the real length and width of the sofa in m.

1. Measure with a ruler. Length on drawing = 2 cm
   Width on drawing = 0.5 cm
2. Multiply to get real-life length. Real length = 2 × 1.5 = 3 m
   Real width = 0.5 × 1.5 = 0.75 m

b) Clare's dining table is 90 cm wide and 180 cm long. Draw the table on the scale drawing.

1. Scale uses m, so convert cm to m. Width = 90 cm = 0.9 m
   Length = 180 cm = 1.8 m
2. Divide to get scale drawing length. Width on drawing = 0.9 ÷ 1.5 = 0.6 cm
   Length on drawing = 1.8 ÷ 1.5 = 1.2 cm
3. Draw with a ruler in any sensible position and label.

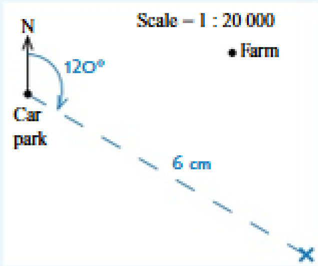


Map Questions Using Bearings

EXAMPLE: Liam walks 1.2 km from the car park on a bearing of 120°.

a) Mark his position on the map.

1. Work out how many km 1 cm represents. 1 cm = 20 000 cm = 200 m = 0.2 km. So 1 cm = 0.2 km
2. Divide to get distance on map. Distance walked on map = 1.2 ÷ 0.2 = 6 cm
3. Mark a point 6 cm away, 120° clockwise from the North line.



b) How far is he from the farm in km?

1. Measure distance between Liam and farm. Distance between Liam and farm = 4 cm
2. For real life, multiply: Real distance = 4 × 0.2 = 0.8 km

THE BIG QUESTION

A map has a scale of 1cm : 3 miles.
On the map, the distance between two towns is 7cm.

What is the actual distance between the two towns?
Include units for your answer.

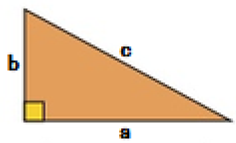
7x3=21

Pythagoras Theorem

Pythagoras' theorem sounds hard but it's actually dead simple.
It's also dead important, so make sure you really get your teeth into it.

Pythagoras' Theorem — $a^2 + b^2 = c^2$

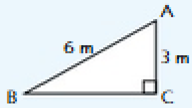
1) PYTHAGORAS' THEOREM only works for RIGHT-ANGLED TRIANGLES.
2) Pythagoras uses two sides to find the third side.
3) The BASIC FORMULA for Pythagoras is $a^2 + b^2 = c^2$
4) Make sure you get the numbers in the RIGHT PLACE. c is the longest side (called the hypotenuse) and it's always opposite the right angle.
5) Always CHECK that your answer is SENSIBLE.



$a^2 + b^2 = c^2$

EXAMPLE:

ABC is a right-angled triangle.
AB = 6 m and AC = 3 m.
Find the exact length of BC.



It's not always c you need to find — loads of people go wrong here.

1) Write down the formula. — $a^2 + b^2 = c^2$
2) Put in the numbers. — $BC^2 + 3^2 = 6^2$
3) Rearrange the equation. — $BC^2 = 6^2 - 3^2 = 36 - 9 = 27$
4) Take square roots to find BC. — $BC = \sqrt{27} = 3\sqrt{3}$ m
5) 'Exact length' means you should give your answer as a surd — simplified if possible.

Remember to check the answer's sensible — here it's about 5.2, which is between 3 and 6, so that seems about right.

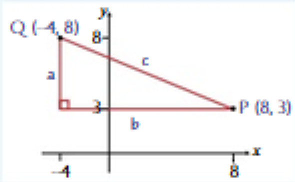
Use Pythagoras to find the Distance Between Points

You need to know how to find the straight-line distance between two points on a graph.
If you get a question like this, follow these rules and it'll all become breathtakingly simple:

1) Draw a sketch to show the right-angled triangle.
2) Find the lengths of the shorter sides of the triangle by subtracting the coordinates.
3) Use Pythagoras to find the length of the hypotenuse. (That's your answer.)

EXAMPLE: Point P has coordinates (8, 3) and point Q has coordinates (–4, 8). Find the length of the line PQ.

①

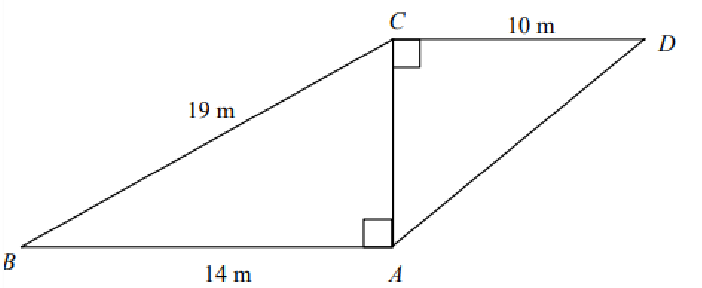


② Length of side a = 8 – 3 = 5
Length of side b = 8 – –4 = 12

③ Use Pythagoras to find side c:
$c^2 = a^2 + b^2 = 5^2 + 12^2 = 25 + 144 = 169$
So: $c = \sqrt{169} = 13$

THE BIG QUESTION

Calculate the length of the *AD*.
Give your answer to 3 significant figures.



$x = \sqrt{265}$
$= 16.3$ (3sf)

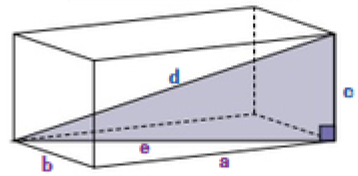
$y = \sqrt{165}$
$= 12.845233...$

This is a 3D version of the 2D Pythagoras theorem you saw on page 95.
There's just one simple formula — learn it and the world's your oyster...

3D Pythagoras for Cuboids — $a^2 + b^2 + c^2 = d^2$ **H**

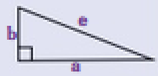
Cuboids have their own formula for calculating the length of their longest diagonal:

$$a^2 + b^2 + c^2 = d^2$$



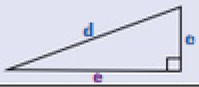
In reality it's nothing you haven't seen before — it's just 2D Pythagoras' theorem being used twice:

1) a, b and e make a right-angled triangle so $e^2 = a^2 + b^2$



2) Now look at the right-angled triangle formed by e, c and d:

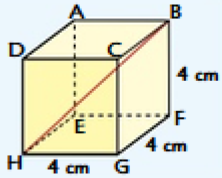
$d^2 = e^2 + c^2 = a^2 + b^2 + c^2$



EXAMPLE:

Find the exact length of the diagonal BH for the cube in the diagram.

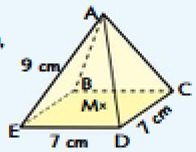
1) Write down the formula. $a^2 + b^2 + c^2 = d^2$
2) Put in the numbers. $4^2 + 4^2 + 4^2 = BH^2$
3) Take the square root to find BH. $\Rightarrow BH = \sqrt{48} = 4\sqrt{3}$ cm



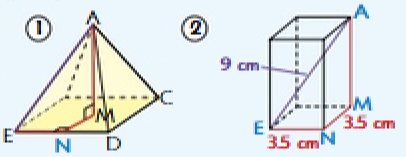
The Cuboid Formula can be used in Other 3D Shapes

EXAMPLE:

In the square-based pyramid shown, M is the midpoint of the base. Find the vertical height AM.



1) Label N as the midpoint of ED.
Then think of EN, NM and AM as three sides of a cuboid, and AE as the longest diagonal in the cuboid (like d in the section above).
2) Sketch the full cuboid.



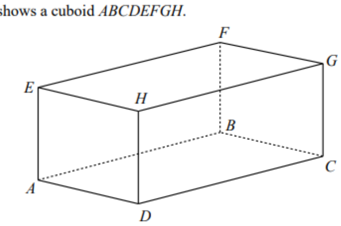
3) Write down the 3D Pythagoras formula. $a^2 + b^2 + c^2 = d^2$
4) Rewrite it using side labels. $EN^2 + NM^2 + AM^2 = AE^2$
5) Put in the numbers and solve for AM. $\Rightarrow 3.5^2 + 3.5^2 + AM^2 = 9^2$

$\Rightarrow AM = \sqrt{81 - 2 \times 12.25} = 7.52$ cm (3 s.f.)

THE BIG QUESTION

The diagram shows a cuboid *ABCDEFGH*.

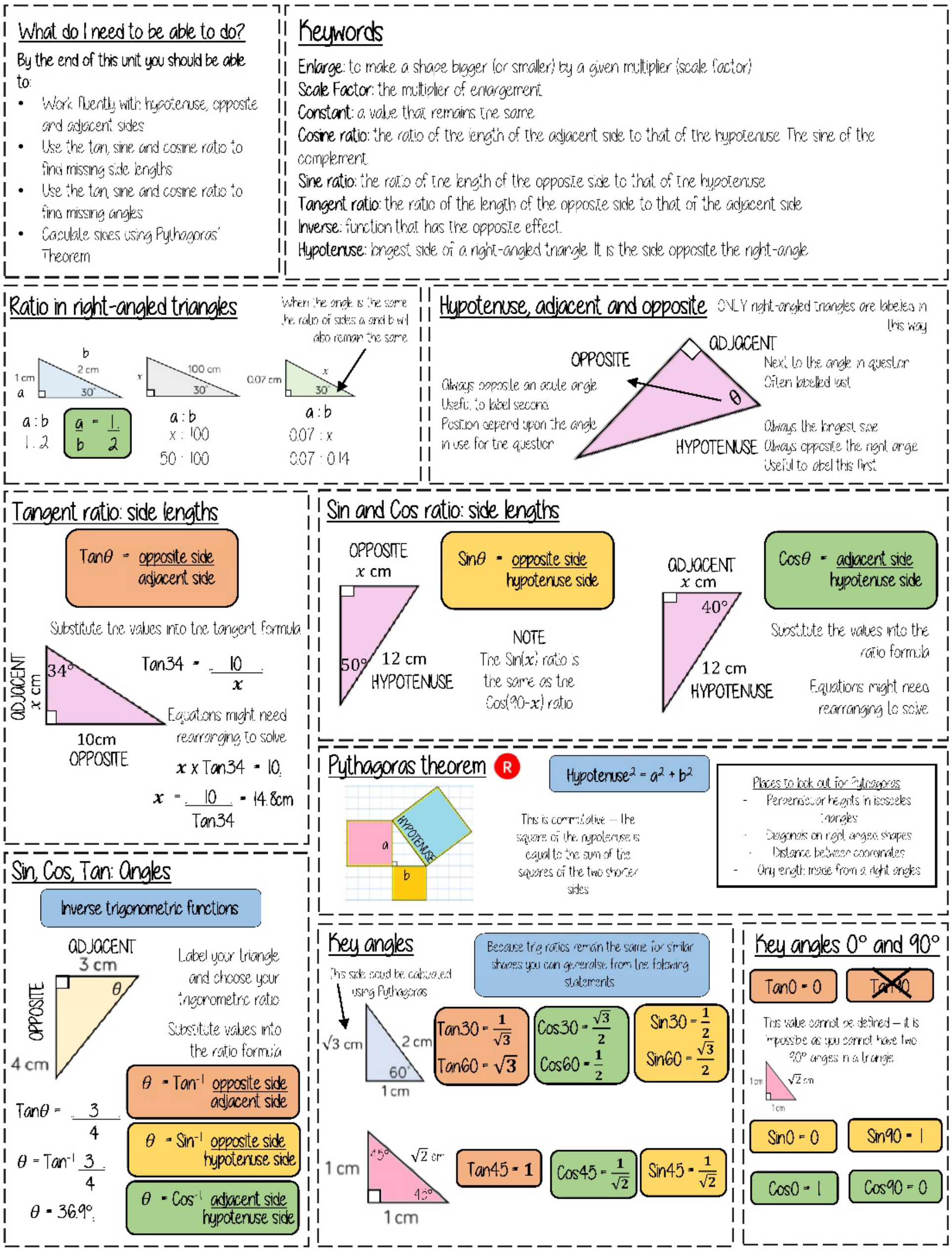
$AE = 4$ cm
$AD = 5$ cm
$DC = 8$ cm



Calculate the length of *AG*.
Give your answer correct to 3 significant figures.

$AG^2 = 5^2 + 8^2 + 4^2$
$= \sqrt{5^2 + 8^2 + 4^2}$
$= 10.2$cm

Trigonometry



What do I need to be able to do?

By the end of this unit you should be able to:

- Work fluently with hypotenuse, opposite and adjacent sides
- Use the tan, sine and cosine ratio to find missing side lengths
- Use the tan, sine and cosine ratio to find missing angles
- Calculate sides using Pythagoras' Theorem

Keywords

Enlarge: to make a shape bigger (or smaller) by a given multiplier (scale factor)
Scale Factor: the multiplier of enlargement
Constant: a value that remains the same
Cosine ratio: the ratio of the length of the adjacent side to that of the hypotenuse. The sine of the complement.
Sine ratio: the ratio of the length of the opposite side to that of the hypotenuse
Tangent ratio: the ratio of the length of the opposite side to that of the adjacent side
Inverse: function that has the opposite effect.
Hypotenuse: longest side of a right-angled triangle. It is the side opposite the right-angle

Ratio in right-angled triangles

When the angle is the same the ratio of sides a and b will also remain the same

a : b = 1 : 2; $\frac{a}{b} = \frac{1}{2}$

a : b; x : 100; 50 : 100

a : b; 0.07 : x; 0.07 : 0.14

Hypotenuse, adjacent and opposite

ONLY right-angled triangles are labelled in this way

OPPOSITE: Always opposite an acute angle. Useful to label second. Position depend upon the angle in use for the question

ADJACENT: Next to the angle in question. Often labelled last

HYPOTENUSE: Always the longest side. Always opposite the right angle. Useful to label this first

Tangent ratio: side lengths

$Tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$

Substitute the values into the tangent formula

$Tan34 = \frac{10}{x}$

Equations might need rearranging to solve

$x \times Tan34 = 10$

$x = \frac{10}{Tan34} = 14.8cm$

Sin and Cos ratio: side lengths

$Sin\theta = \frac{\text{opposite side}}{\text{hypotenuse side}}$

$Cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse side}}$

NOTE: The Sin(x) ratio is the same as the Cos(90-x) ratio

Substitute the values into the ratio formula

Equations might need rearranging to solve

Pythagoras theorem (R)

$Hypotenuse^2 = a^2 + b^2$

This is commutative – the square of the hypotenuse is equal to the sum of the squares of the two shorter sides

Places to look out for Pythagoras

- Perpendicular heights in isosceles triangles
- Diagonals on rigid angled shapes
- Distance between coordinates
- Any length made from 2 right angles

Sin, Cos, Tan: Angles

Inverse trigonometric functions

Label your triangle and choose your trigonometric ratio

Substitute values into the ratio formula

$\theta = Tan^{-1} \frac{\text{opposite side}}{\text{adjacent side}}$

$\theta = Sin^{-1} \frac{\text{opposite side}}{\text{hypotenuse side}}$

$\theta = Cos^{-1} \frac{\text{adjacent side}}{\text{hypotenuse side}}$

$Tan\theta = \frac{3}{4}$

$\theta = Tan^{-1}\frac{3}{4}$

$\theta = 36.9°$

Key angles

Because trig ratios remain the same for similar shapes you can generalise from the following statements

This side could be calculated using Pythagoras

$Tan30 = \frac{1}{\sqrt{3}}$, $Tan60 = \sqrt{3}$

$Cos30 = \frac{\sqrt{3}}{2}$, $Cos60 = \frac{1}{2}$

$Sin30 = \frac{1}{2}$, $Sin60 = \frac{\sqrt{3}}{2}$

$Tan45 = 1$, $Cos45 = \frac{1}{\sqrt{2}}$, $Sin45 = \frac{1}{\sqrt{2}}$

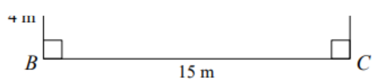
Key angles 0° and 90°

Tan0 = 0; ~~Tan90~~

This value cannot be defined – it is impossible as you cannot have two 90° angles in a triangle

Sin0 = 0; Sin90 = 1

Cos0 = 1; Cos90 = 0



Work out the size of angle *BAD*.
Give your answer to 1 decimal place.

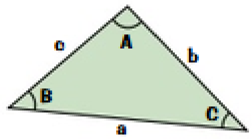
Sine and Cosine rule

Normal trigonometry using SOH CAH TOA etc. can only be applied to right-angled triangles. Which leaves us with the question of what to do with other-angled triangles. Step forward the Sine and Cosine Rules...

Labelling the Triangle

H

This is very important. You must label the sides and angles properly so that the letters for the sides and angles correspond with each other. Use lower case letters for the sides and capitals for the angles.



Remember, side 'a' is opposite angle A etc.

It doesn't matter which sides you decide to call a, b, and c, just as long as the angles are then labelled properly.

Three Formulas to Learn:

The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

You don't use the whole thing with both '=' signs of course, so it's not half as bad as it looks — you just choose the two bits that you want:

e.g. $\frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{a}{\sin A} = \frac{b}{\sin B}$

The Cosine Rule

The 'normal' form is...

$$a^2 = b^2 + c^2 - 2bc \cos A$$

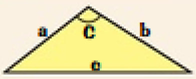
...or this form is good for finding an angle (you get it by rearranging the 'normal' version):

$$\text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Area of the Triangle

This formula comes in handy when you know two sides and the angle between them:

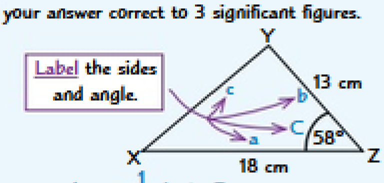
$$\text{Area of triangle} = \tfrac{1}{2} ab \sin C$$



Of course, you already know a simple formula for calculating the area using the base length and height (see p82). The formula here is for when you don't know those values.

EXAMPLE:

Triangle XYZ has XZ = 18 cm, YZ = 13 cm and angle XZY = 58°. Find the area of the triangle, giving your answer correct to 3 significant figures.



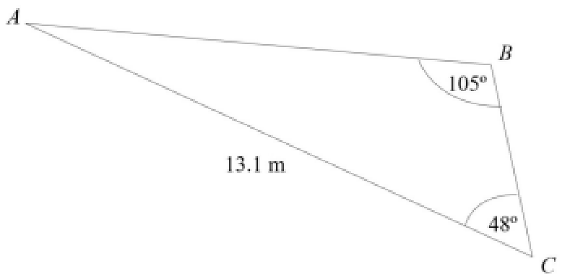
$\text{Area} = \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 18 \times 13 \times \sin 58°$

$= 99.2\ \text{cm}^2$ (3 s.f.)

Don't forget the units.

THE BIG QUESTION



Work out the perimeter of triangle *ABC*.
Give your answer to 3 significant figures.

$\frac{\sin(105)}{13.1} = \frac{\sin(48)}{x}$

$x = \frac{\sin(105)}{13.1} \times \sin(48)$

$= 10.0786177(9)$

$\frac{\sin(105)}{13.1} = \frac{\sin(27)}{y}$

$y = \frac{\sin(105)}{13.1} \times \sin(27)$

$= 6.1570727(12)$

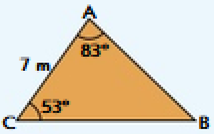
$x + y + 13.1 = 29.3$ m (3sf)

There are four main question types where the sine and cosine rules would be applied. So learn the exact details of these four examples and you'll be laughing. **WARNING:** if you laugh too much people will think you're crazy.

The Four Examples **H**

1 TWO ANGLES given plus ANY SIDE — SINE RULE needed.

Find the length of AB for the triangle below.



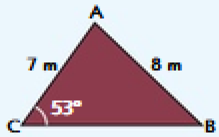
1) Don't forget the obvious... $B = 180° - 83° - 53° = 44°$

2) Put the numbers into the sine rule. $\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{7}{\sin 44°} = \frac{c}{\sin 53°}$

3) Rearrange to find c. $\Rightarrow c = \frac{7 \times \sin 53°}{\sin 44°} = 8.05$ m (3 s.f.)

2 TWO SIDES given plus an ANGLE NOT ENCLOSED by them — SINE RULE needed.

Find angle ABC for the triangle shown below.



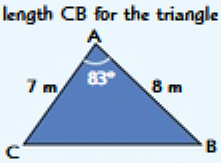
1) Put the numbers into the sine rule. $\frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{7}{\sin B} = \frac{8}{\sin 53°}$

2) Rearrange to find sin B. $\Rightarrow \sin B = \frac{7 \times \sin 53°}{8} = 0.6988...$

3) Find the inverse. $\Rightarrow B = \sin^{-1}(0.6988...) = 44.3°$ (1 d.p.)

3 TWO SIDES given plus the ANGLE ENCLOSED by them — COSINE RULE needed.

Find the length CB for the triangle shown below.



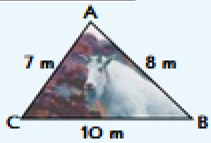
1) Put the numbers into the cosine rule. $a^2 = b^2 + c^2 - 2bc \cos A$
$= 7^2 + 8^2 - 2 \times 7 \times 8 \times \cos 83°$
$= 99.3506...$

2) Take square roots to find a. $a = \sqrt{99.3506...}$
$= 9.97$ m (3 s.f.)

You might come across a triangle that isn't labelled ABC — just relabel it yourself to match the sine and cosine rules.

4 ALL THREE SIDES given but NO ANGLES — COSINE RULE needed.

Find angle CAB for the triangle shown.

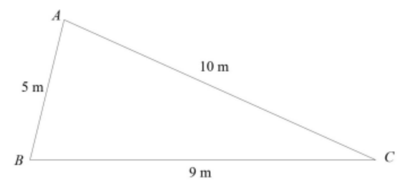


1) Use this version of the cosine rule. $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

2) Put in the numbers. $= \frac{49 + 64 - 100}{2 \times 7 \times 8}$

3) Take the inverse to find A. $= \frac{13}{112} = 0.11607...$
$\Rightarrow A = \cos^{-1}(0.11607...)$
$= 83.3°$ (1 d.p.)

THE BIG QUESTION



Work out the size of angle *ABC*
Give your answer to the nearest degree (3 marks)

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\cos x = \frac{(15)^2 + (16)^2 - (20)^2}{2(15)(16)}$

$\cos x = \frac{27}{160}$

$x = \cos^{-1}$ (ans)

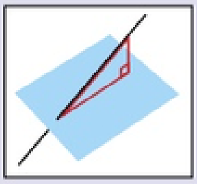
$= 80.3$ (3sf)

3D trig may sound tricky, and I suppose it is a bit... but it's actually just using the same old rules.

Angle Between Line and Plane — Use a Diagram **H**

Learn the 3-Step Method

1) Make a right-angled triangle between the line and the plane.

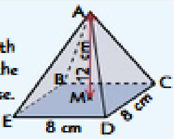


2) Draw a simple 2D sketch of this triangle and mark on the lengths of two sides (you might have to use Pythagoras to find one).
3) Use trig to find the angle.

Have a look at p.95-98 to jog your memory about Pythagoras and trig.

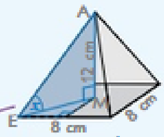
EXAMPLE:

ABCDE is a square-based pyramid with M as the midpoint of its base. Find the angle the edge AE makes with the base.

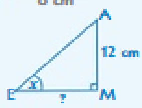


1) Draw a right-angled triangle using AE, the base and a line between the two (here it's the vertical height).

Label the angle you need to find.



2) Now sketch this triangle in 2D and label it.



Use Pythagoras (on the base triangle) to find EM.

$EM^2 = 4^2 + 4^2 = 32$

$\rightarrow EM = \sqrt{32}$ cm

3) Finally, use trigonometry to find x — you know the opposite and adjacent sides so use tan.

$\tan x = \frac{12}{\sqrt{32}} = 2.1213...$

$x = \tan^{-1}(2.1213...)$

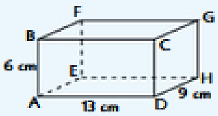
$= 64.8°$ (1 d.p.)

The Sine Rule and Cosine Rule can also be used in 3D

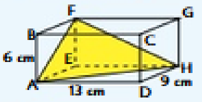
For triangles inside 3D shapes that aren't right-angled you can use the sine and cosine rules. This sounds mildly terrifying but it's actually OK — just use the same formulas as before (see p.99-100).

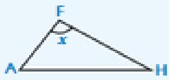
EXAMPLE:

Find the size of angle AFH in the cuboid shown below.



1) Draw the triangle AFH and label angle AFH as x.





2) Use Pythagoras' theorem to find the lengths of AF, AH and FH.

$AH^2 = 13^2 + 9^2 = 250 \Rightarrow AH = \sqrt{250}$

$AF^2 = 6^2 + 9^2 = 117 \Rightarrow AF = \sqrt{117}$

$FH^2 = 6^2 + 13^2 = 205 \Rightarrow FH = \sqrt{205}$

3) Find x using the cosine rule: — $AH^2 = AF^2 + FH^2 - 2 \times AF \times FH \times \cos x$

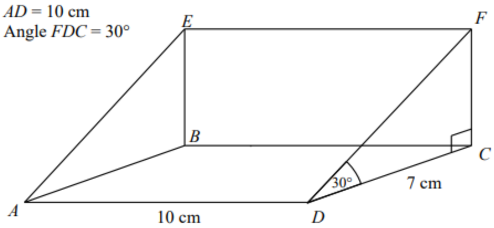
Put in the numbers. — $250 = 117 + 205 - 2\sqrt{117}\sqrt{205}\cos x$

Rearrange and take the inverse to find x. — $x = \cos^{-1}\left(\frac{117 + 205 - 250}{2\sqrt{117 \times 205}}\right) = 76.6°$ (1 d.p.)

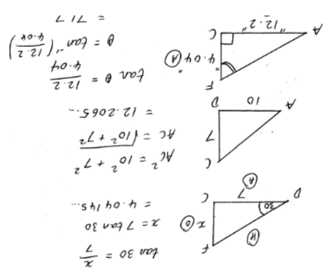
THE BIG QUESTION

The diagram shows a triangular prism.

$CD = 7$ cm
$AD = 10$ cm
Angle $FDC = 30°$



Calculate the size of angle AFC.
Give your answer correct to 1 decimal place.



Ratio

Ratios are a pretty important topic — so work your way through the examples on the next three pages, and the whole murky business should become crystal clear...

Writing Ratios as Fractions

This is a simple one — to write a ratio as a fraction just put one number over the other.

E.g. if apples and oranges are in the ratio 2:9 then we say there are $\frac{2}{9}$ as many apples as oranges or $\frac{9}{2}$ times as many oranges as apples.

Reducing Ratios to their Simplest Form

To reduce a ratio to a simpler form, divide all the numbers in the ratio by the same thing (a bit like simplifying a fraction — see p.5). It's in its simplest form when there's nothing left you can divide by.

EXAMPLE: Write the ratio 15:18 in its simplest form.

For the ratio 15:18, both numbers have a factor of 3, so divide them by 3.

We can't reduce this any further. So the simplest form of 15:18 is 5:6.

÷3 (15:18) ÷3
= 5:6

A handy trick for the calculator papers — use the fraction button

If you enter a fraction with the [fraction] or [a b/c] button, the calculator automatically cancels it down when you press [=].

So for the ratio 8:12, just enter $\frac{8}{12}$ as a fraction, and you'll get the reduced fraction $\frac{2}{3}$.

Now you just change it back to ratio form, i.e. 2 : 3. Ace.

The More Awkward Cases:

1) If the ratio contains decimals or fractions — multiply

For fractions, multiply by a number that gets rid of both denominators.

EXAMPLE: Simplify the ratio 2.4:3.6 as far as possible.

1) Multiply both sides by 10 to get rid of the decimal parts.
2) Now divide to reduce the ratio to its simplest form.

×10 (2.4:3.6) ×10
= 24:36
÷12 ( ) ÷12
= 2:3

2) If the ratio has mixed units — convert to the smaller unit

EXAMPLE: Reduce the ratio 24 mm:7.2 cm to its simplest form.

1) Convert 7.2 cm to millimetres.
2) Simplify the resulting ratio. Once the units on both sides are the same, get rid of them for the final answer.

24 mm:7.2 cm
= 24 mm:72 mm
÷24 ( ) ÷24
= 1:3

3) To get to the form 1 : n or n : 1 — just divide

This form is often the most useful, since it shows the ratio very clearly.

EXAMPLE: Reduce 3:56 to the form 1:n.

Divide both sides by 3:

÷3 (3:56) ÷3
= $1:\frac{56}{3}$ = $1:18\frac{2}{3}$ (or $1:18.\dot{6}$)

Another page on ratios coming up — it's more interesting than the first but not as exciting as the next one...

Scaling Up Ratios

If you know the ratio between parts and the actual size of one part, you can scale the ratio up to find the other parts.

EXAMPLE: Mortar is made from mixing sand and cement in the ratio 7:2. How many buckets of mortar will be made if 21 buckets of sand are used in the mixture?

You need to multiply by 3 to go from 7 to 21 on the left-hand side (LHS) — so do that to both sides:

sand:cement
×3 (7:2) ×3
= 21:6

So 21 buckets of sand and 6 buckets of cement are used.

Amount of mortar made = 21 + 6 = 27 buckets

The two parts of a ratio are always in direct proportion (see p.62). So in the example above, sand and cement are in direct proportion, e.g. if the amount of sand doubles, the amount of cement doubles.

Part : Whole Ratios

You might come across a ratio where the LHS is included in the RHS — these are called part : whole ratios.

EXAMPLE: Mrs Miggins owns tabby cats and ginger cats. The ratio of tabby cats to the total number of cats is 3:5.

a) What fraction of Mrs Miggins' cats are tabby cats?

The ratio tells you that for every 5 cats, 3 are tabby cats.

$\frac{\text{part}}{\text{whole}} = \frac{3}{5}$

b) What is the ratio of tabby cats to ginger cats?

3 in every 5 cats are tabby, so 2 in every 5 are ginger.

5 − 3 = 2

For every 3 tabby cats there are 2 ginger cats.

tabby:ginger = 3:2

c) Mrs Miggins has 12 tabby cats. How many ginger cats does she have?

Scale up the ratio from part b) to find the number of ginger cats.

tabby:ginger
×4 (3:2) ×4
= 12:8

There are 8 ginger cats

Proportional Division

In a proportional division question a TOTAL AMOUNT is split into parts in a certain ratio.
The key word here is PARTS — concentrate on 'parts' and it all becomes quite painless:

EXAMPLE: Jess, Mo and Greg share £9100 in the ratio 2:4:7. How much does Mo get?

1) ADD UP THE PARTS:
The ratio 2:4:7 means there will be a total of 13 parts: 2 + 4 + 7 = 13 parts

2) DIVIDE TO FIND ONE "PART":
Just divide the total amount by the number of parts: £9100 ÷ 13 = £700 (= 1 part)

3) MULTIPLY TO FIND THE AMOUNTS:
We want to know Mo's share, which is 4 parts: 4 parts = 4 × £700 = £2800

If you were worried I was running out of great stuff to say about ratios then worry no more...

Changing Ratios

You'll need to know how to deal with all sorts of questions where the ratio changes.
Have a look at the examples to see how to handle them.

EXAMPLE: In an animal sanctuary there are 20 peacocks, and the ratio of peacocks to pheasants is 4:9. If 5 of the pheasants fly away, what is the new ratio of peacocks to pheasants? Give your answer in its simplest form.

1) Find the original number of pheasants.

peacocks:pheasants

$4:9$ (×5) → $20:45$

2) Work out the number of pheasants remaining.

$45 - 5 = 40$ pheasants left

3) Write the new ratio of peacocks to pheasants and simplify.

peacocks:pheasants

$20:40$ (÷20) → $1:2$

EXAMPLE: **H**

The ratio of male to female pupils going on a skiing trip is 5:3. Four male teachers and nine female teachers are also going on the trip. The ratio of males to females going on the trip is 4:3 (including teachers). How many female pupils are going on the trip?

1) WRITE THE RATIOS AS EQUATIONS

Let m be the number of male pupils and f be the number of female pupils.

$m:f = 5:3$

$(m + 4):(f + 9) = 4:3$

2) TURN THE RATIOS INTO FRACTIONS (see p.59)

$\frac{m}{f} = \frac{5}{3}$ and $\frac{m+4}{f+9} = \frac{4}{3}$

$3m = 5f$ and $3m + 12 = 4f + 36$

3) SOLVE THE TWO EQUATIONS SIMULTANEOUSLY.

$$3m - 4f = 24$$
$$-\quad 3m - 5f = 0$$
$$f = 24$$

See pages 37-38 for more on simultaneous equations.

24 female pupils are going on the trip.

THE BIG QUESTION

tion 29: Class 10D make some cakes using milk chocolate, dark chocolate or white chocolate.
Some of the cakes contain nuts and the rest do not.

The ratio of the number of milk chocolate cakes to dark chocolate cakes is 10:3
The ratio of the number of white chocolate cakes to milk chocolate cakes is 1:6

Of the milk chocolate cakes, the ratio of the number of cakes containing nuts to not containing nuts is 1:8

Of the dark chocolate cakes, the ratio of the number of cakes containing nuts to not containing nuts is 3:2

Of the white chocolate cakes, the ratio of the number of cakes containing nuts to not containing nuts is 2:5

What percentage of the cakes contain nuts?

23.1%

Rounding and estimation

There are two different ways of specifying where a number should be rounded.
They are: 'Decimal Places' and 'Significant Figures'.

Decimal Places (d.p.)

To round to a given number of decimal places:

1) IDENTIFY the position of the 'LAST DIGIT' from the number of decimal places.
2) Then look at the next digit to the RIGHT — called THE DECIDER.
3) If the DECIDER is 5 OR MORE, then ROUND UP the LAST DIGIT.
   If the DECIDER is 4 OR LESS, then LEAVE the LAST DIGIT as it is.
4) There must be NO MORE DIGITS after the last digit (not even zeros).

'Last digit' = last one in the rounded version, not the original number.

EXAMPLE: What is 7.45839 to 2 decimal places?

7.45839 = 7.46

LAST DIGIT: 5; DECIDER: 8

The LAST DIGIT rounds UP because the DECIDER is 5 or more.

If you have to round up a 9 (to 10), replace the 9 with 0, and carry 1 to the left.
Remember to keep enough zeros to fill the right number of decimal places — so to 2 d.p. 45.699 would be rounded to 45.70, and 64.996 would be rounded to 65.00.

65 has the same value as 65.00, but 65 isn't expressed to 2 d.p. so it would be marked wrong.

Significant Figures (s.f.)

The method for significant figures is identical to that for decimal places except that locating the last digit is more difficult — it wouldn't be so bad, but for the zeros...

1) The 1st significant figure of any number is simply the first digit which isn't a zero.

2) The 2nd, 3rd, 4th, etc. significant figures follow on immediately after the 1st, regardless of being zeros or not zeros.

0.002309 — SIG. FIGS: 2 (1st), 3 (2nd), 0 (3rd), 9 (4th)

2.03070 — SIG. FIGS: 2 (1st), 0 (2nd), 3 (3rd), 0 (4th)

(If we're rounding to say, 3 s.f., then the LAST DIGIT is simply the 3rd sig. fig.)

3) After rounding the last digit, end zeros must be filled in up to, but not beyond, the decimal point.

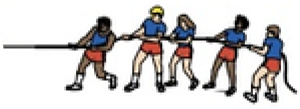
No extra zeros must ever be put in after the decimal point.

EXAMPLES:

		to 3 s.f.	to 2 s.f.	to 1 s.f.
1)	54.7651	54.8	55	50
2)	0.0045902	0.00459	0.0046	0.005
3)	30895.4	30900	31000	30000

'Estimating' doesn't mean 'take a wild guess', it means 'look at the numbers, make them a bit easier, then do the calculation'. Your answer won't be as accurate as the real thing but hey, it's easier on your brain.

Estimating Calculations



It's time to put your rounding skills to use and do some estimating.

EXAMPLE: Estimate the value of $\frac{127.8 + 41.9}{56.5 \times 3.2}$, showing all your working.

1) Round all the numbers to easier ones — 1 or 2 s.f. usually does the trick.
2) You can round again to make later steps easier if you need to.

$$\frac{127.8 + 41.9}{56.5 \times 3.2} \approx \frac{130 + 40}{60 \times 3} = \frac{170}{180} \approx 1$$

EXAMPLE: A cylindrical glass has a height of 18 cm and a radius of 3 cm.
a) Find an estimate in cm³ for the volume of the glass.

The formula for the volume of a cylinder is $V = \pi r^2 h$ (see p.85).

Round the numbers to 1 s.f.:

$\pi = 3.14159... = 3$ (1 s.f.), height = 20 cm (1 s.f.) and radius = 3 cm (1 s.f.).

Now just put the numbers into the formula:

$V = \pi r^2 h \approx 3 \times 3^2 \times 20 = 3 \times 9 \times 20 \approx 540 \text{ cm}^3$

≈ means 'approximately equal to'.

b) Use your answer to part a) to estimate the number of glasses that could be filled from a 2.5 litre bottle of lemonade.

2.5 litres = 2500 cm³

$2500 \div 540 \approx 2500 \div 500 = 5$ glasses

The number of glasses must be an integer.

Estimating Square Roots

Estimating square roots can be a bit tricky, but there are only 2 steps:

1) Find two square numbers, one either side of the number you're given.
2) Decide which number it's closest to, and make a sensible estimate of the digit after the decimal point.

EXAMPLE: Estimate the value of $\sqrt{87}$ to 1 d.p.

87 is between 81 (= 9²) and 100 (= 10²).

It's closer to 81, so its square root will be closer to 9 than 10: $\sqrt{87} \approx 9.3$

(the actual value of $\sqrt{87}$ is 9.32737..., so this is a reasonable estimate).

THE BIG QUESTION

Eddie and Ellen use a calculator to work out $\frac{431.1}{14.3 + 3.8^2}$

Eddie's answer is 1.5
Ellen's answer is 15

One of those answers is correct.

Use approximations to find out which answer is correct. **(3 marks)**

Ellen's answer is correct

$= 15.3$

$\frac{14+16}{400} = \frac{30}{40}$

$\frac{14+4^2}{400}$ $\left[\frac{10+4}{400}\right]$

Error intervals and bounds

Whenever a number is rounded or truncated it will have some amount of error.
The error tells you how far the actual value could be away from the rounded value.

Rounded Measurements Can Be Out By Half A Unit

Whenever a measurement is rounded off to a given UNIT the actual measurement can be anything up to HALF A UNIT bigger or smaller.

EXAMPLES:

1. A room is measured to be 9 m long to the nearest metre. What are its minimum and maximum possible lengths?

The measurement is to the nearest 1 m, so the actual length could be up to 0.5 m bigger or smaller.

Minimum length = 9 – 0.5 = 8.5 m
Maximum length = 9 + 0.5 = 9.5 m

The actual maximum length is 9.4999... m, but it's OK to say 9.5 m instead.

If you're asked for the error interval, you can use inequalities to show the actual maximum:

2. The mass of a cake is given as 2.4 kg to the nearest 0.1 kg. Find the interval within which m, the actual mass of the cake, lies.

Minimum mass = 2.4 – 0.05 = 2.35 kg
Maximum mass = 2.4 + 0.05 = 2.45 kg

So the interval is $2.35 \text{ kg} \le m < 2.45 \text{ kg}$

See p.37 for more on inequalities.

The actual value is greater than or equal to the minimum but strictly less than the maximum.
The actual mass of the cake could be exactly 2.35 kg, but if it was exactly 2.45 kg it would round up to 2.5 kg instead.

Truncated Measurements Can Be A Whole Unit (

You truncate a number by chopping off decimal places. E.g. 25.765674 truncated to 1 d.p. would be 25.7

When a measurement is TRUNCATED to a given UNIT, the actual measurement can be up to A WHOLE UNIT bigger but no smaller.

If the mass of the cake in example 2 was 2.4 kg truncated to 1 d.p. the error interval would be $2.4 \text{ kg} \le m < 2.5 \text{ kg}$.
So even if the mass was 2.499999 kg, it would still truncate to 2.4 kg.

THE BIG QUESTION

The weight of a bag of potatoes is 15 kg, correct to the nearest kg.

(a) Write down the smallest possible weight of the bag of potatoes.

(b) Write down the largest possible weight of the bag of potatoes.

14.5 kg (1)

15.5 kg (1)

Finding upper and lower bounds is pretty easy, but using them in calculations is a bit trickier.

Upper and Lower Bounds

When a measurement is ROUNDED to a given UNIT, the actual measurement can be anything up to HALF A UNIT bigger or smaller.

EXAMPLE: The mass of a cake is given as 2.4 kg to the nearest 0.1 kg. Find the interval within which m, the actual mass of the cake, lies.

See p.33 for more on inequalities.

lower bound = 2.4 – 0.05 = 2.35 kg
upper bound = 2.4 + 0.05 = 2.45 kg

So the interval is $2.35 \text{ kg} \le m < 2.45 \text{ kg}$

The actual value is greater than or equal to the lower bound but strictly less than the upper bound. The actual mass of the cake could be exactly 2.35 kg, but if it was exactly 2.45 kg it would round up to 2.5 kg instead.

When a measurement is TRUNCATED to a given UNIT, the actual measurement can be up to A WHOLE UNIT bigger but no smaller.

You truncate a number by chopping off decimal places, so if the mass of the cake was 2.4 truncated to 1 d.p. the interval would be $2.4 \text{ kg} \le x < 2.5 \text{ kg}$.

If the mass was 2.49999, it would still be truncated to 2.4.

Maximum and Minimum Values for Calculations

H

When a calculation is done using rounded values there will be a DISCREPANCY between the CALCULATED VALUE and the ACTUAL VALUE:

EXAMPLES:

1. A pinboard is measured as being 0.89 m wide and 1.23 m long, to the nearest cm.
   a) Calculate the minimum and maximum possible values for the area of the pinboard.

Find the bounds for the width and length:
$0.885 \text{ m} \le \text{width} < 0.895 \text{ m}$
$1.225 \text{ m} \le \text{length} < 1.235 \text{ m}$

Find the minimum area by multiplying the lower bounds, and the maximum by multiplying the upper bounds:

minimum possible area $= 0.885 \times 1.225 = 1.084125 \text{ m}^2$

maximum possible area $= 0.895 \times 1.235 = 1.105325 \text{ m}^2$

b) Use your answers to part a) to give the area of the pinboard to an appropriate degree of accuracy.
The area of the pinboard lies in the interval $1.084125 \text{ m}^2 \le a < 1.105325 \text{ m}^2$. Both the upper bound and the lower bound round to 1.1 m^2 to 1 d.p. so the area of the pinboard is 1.1 m^2 to 1 d.p.

2. $a = 5.3$ and $b = 4.2$, both given to 1 d.p. What are the maximum and minimum values of $a \div b$?

First find the bounds for a and b. → $5.25 \le a < 5.35$, $4.15 \le b < 4.25$

Now the tricky bit... The bigger the number you divide by, the smaller the answer, so:
$\max(a \div b) = \max(a) \div \min(b)$
and $\min(a \div b) = \min(a) \div \max(b)$

max. value of $a \div b = 5.35 \div 4.15 = 1.289$ (to 3 d.p.)
min. value of $a \div b = 5.25 \div 4.25 = 1.235$ (to 3 d.p.)

THE BIG QUESTION

A rectangle has a length of 21cm, to the nearest cm, and a width of 5.3cm, to the nearest mm.

a) Work out the upper bound for the perimeter of the rectangle.

b) Work out the lower bound for the area of the rectangle.

107.625

53.7

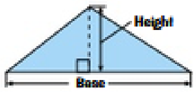
Area, surface area, and volume of shapes

Be warned — there are lots of area formulas coming up on the next two pages for you to learn. By the way, I'm assuming that you know the formulas for the area of a rectangle (A = l × w) and the area of a square (A = l²).

Areas of Triangles and Quadrilaterals

LEARN these Formulas:

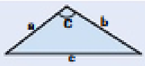
Note that in each case the height must be the vertical height, not the sloping height.



Area of triangle = ½ × base × vertical height

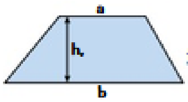
$A = \frac{1}{2} \times b \times h_v$

The alternative formula is:
Area of triangle = ½ ab sin C
This is covered on p.99.



Area of parallelogram = base × vertical height

$A = b \times h_v$



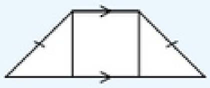
Area of trapezium = average of parallel sides × distance between them (vertical height)

$A = \frac{1}{2}(a + b) \times h_v$

Use the Formulas to Solve Problems

Examiners like to sneak bits of algebra into area and perimeter questions — you'll often have to set up and then solve an equation to find a missing side length or area of a shape. Meanies.

EXAMPLE: The shape on the right shows a square with sides of length x cm drawn inside an isosceles trapezium. The base of the trapezium is three times as long as one side of the square.



In an isosceles trapezium, the sloping sides are the same length.

a) Find an expression for the area of the trapezium in terms of x.

Top of trapezium = side of square = x cm
Base of trapezium = 3 × side of square = $3x$ cm
Height of trapezium = side of square = x cm
Area of trapezium = $\frac{1}{2}(x + 3x) \times x = 2x^2$ cm²

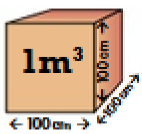
b) The area of the trapezium is 60.5 cm². Find the side length of the square.

Set your equation from part a) equal to 60.5 and solve to find x:

$2x^2 = 60.5$
$x^2 = 30.25$
$x = 5.5$ cm

Converting Volumes

1 m³ = 100 cm × 100 cm × 100 cm = 1 000 000 cm³
1 cm³ = 10 mm × 10 mm × 10 mm = 1000 mm³



1) Find the conversion factor — it'll be the same as for converting units (see p66).
2) Multiply AND divide by the conversion factor THREE TIMES.
3) Choose the common sense answer, and don't forget that the units come with a 3, e.g. mm³, cm³.

EXAMPLE: A glass has a volume of 72 000 mm³. What is its volume in cm³?

1) Find the conversion factor: 1 cm = 10 mm ⟶ Conversion factor = 10
2) It's a volume — multiply and divide 3 times by the conversion factor: ~~72 000 × 10 × 10 × 10 = 72 000 000~~
72 000 ÷ 10 ÷ 10 ÷ 10 = 72
3) Choose the sensible answer: 72 000 mm³ = 72 cm³

1 cm = 10 mm, so expect fewer cm than mm.

What do I need to be able to do?

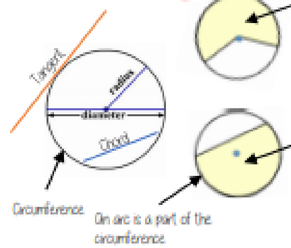
By the end of this unit you should be able to:

- Recognise and label parts of a circle
- Calculate fractional parts of a circle
- Calculate the length of an arc
- Calculate the area of a sector
- Understand and use volume of a cone, cylinder and sphere
- Understand and use surface area of a cone, cylinder and sphere

Keywords

Circumference: the length around the outside of the circle – the perimeter
Area: the size of the 2D surface
Diameter: the distance from one side of a circle to another through the centre
Radius: the distance from the centre to the circumference of the circle
Tangent: a straight line that touches the circumference of a circle
Chord: a line segment connecting two points on the curve
Frustum: a pyramid or cone with the top cut off
Hemisphere: half a sphere
Surface area: the total area of the surface of a 3D shape

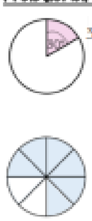
Parts of a circle (R)



Fractional parts of a circle

A circle is made up of 360°

Formula to remember
Area of a circle $= \pi r^2$
Circumference of a circle $= \pi d$ or $2\pi r$



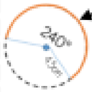
30° represents $\frac{30}{360}$ of a full circle → $\frac{30}{360} = \frac{1}{12}$

$\frac{270}{360}$ of a full circle (in degrees)
$\frac{6}{8}$ of a full circle (in equal parts)
$\frac{3}{4}$ of a full circle

The fraction of the circle is $\frac{\theta}{360}$

θ represents the degrees in the sector

Arc length



Remember an arc is part of the circumference
Circumference of the whole circle $= \pi d = \pi \times 9 = 9\pi$

Arc length $= \frac{\theta}{360} \times$ circumference

$= \frac{240}{360} \times 9\pi$

$= \frac{2}{3} \times 9\pi = \underline{6\pi}$

Perimeter

Perimeter is the length around the outside of the shape
This includes the arc length and the radii that encloses the shape

Perimeter $= \frac{\theta}{360} \times$ circumference $+ 2r$ $= \underline{6\pi + 9}$

Sector area

Remember a sector is part of a circle
Area of the whole circle $= \pi r^2 = \pi \times 6^2 = 36\pi$

Sector area $= \frac{\theta}{360} \times$ area of circle

$= \frac{120}{360} \times 36\pi$

$= \frac{1}{3} \times 36\pi = \underline{12\pi}$

Volume of a sphere

Volume Sphere $= \frac{4}{3}\pi r^3$

NOTE: This is now a **cubed** value

Volume Sphere $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times 3^3$

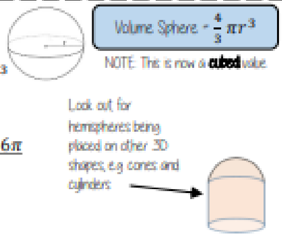
$= \frac{4}{3} \times \pi \times 27 = \underline{36\pi}$

A hemisphere is half the volume of the overall sphere

$= 36\pi \div 2$

$= 18\pi$

Look out for hemispheres being placed on other 3D shapes, e.g. cones and cylinders



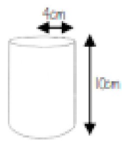
Volume of a cone and a cylinder

Volume Cylinder $= \pi r^2 h$

A cylinder is a prism – cross section is a circle

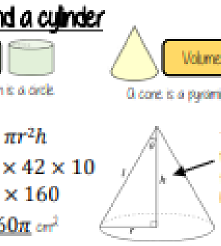
Volume Cone $= \frac{1}{3}\pi r^2 h$

A cone is a pyramid with a circular base



$V = \pi r^2 h$
$= \pi \times 42 \times 10$
$= \pi \times 160$
$= \underline{160\pi}$ cm³

Give your answer in terms of π' means **NOT** in terms of pi $= \underline{502.7}$cm³

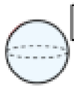


The height of a cone is the perpendicular height from the vertex to the base

Look out for trigonometry or Pythagoras theorem – the radius forms the base of a right-angled triangle

Surface area of a sphere

Surface area $= 4\pi r^2$



Radius = 5cm

Surface area $= 4\pi r^2$

$= 4 \times \pi \times 5^2$

$= 4 \times \pi \times 25$

$= \underline{100\pi}$

The curved surface area of a sphere

A hemisphere has the curved surface AND a flat circular face

$= 100\pi \div 2 = 50\pi$

$= 50\pi + \pi \times 5^2$

Hemisphere $= \underline{75\pi}$

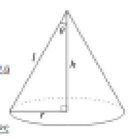
Surface area of cones and cylinders

Surface area cylinder $= 2\pi r^2 + \pi d h$

The area of two circles (top and bottom face) + the area of the curved face

The length of shape B is the circumference of the circles

Curved surface area Cone $= \pi r l$

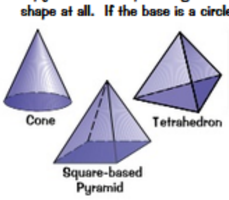


Look out for the use of Pythagoras to calculate the length l

Total surface area = curved face + circle face (area of base)

Volumes of Pyramids and Cones

A pyramid is a shape that goes from a flat base up to a point at the top. Its base can be any shape at all. If the base is a circle then it's called a cone (rather than a circular pyramid).



VOLUME OF PYRAMID $= \frac{1}{3} \times$ BASE AREA $\times$ VERTICAL HEIGHT

VOLUME OF CONE $= \frac{1}{3} \times \pi r^2 \times h_v$

Make sure you use the vertical (perpendicular) height in these formulas — don't get confused with the slant height, which you used to find the surface area of a cone.

Another page on volumes now, but this is a bit of a weird one.
First up, it's volumes of cones with a bit chopped off, then it's on to rates of flow.

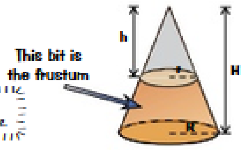
Volumes of Frustums H

A frustum of a cone is what's left when the top part of a cone is cut off parallel to its circular base.

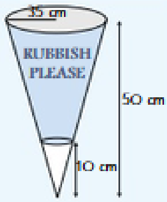
VOLUME OF FRUSTUM = VOLUME OF ORIGINAL CONE − VOLUME OF REMOVED CONE

$$= \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h$$

The bit that's chopped off is a mini cone that's similar to the original cone.



EXAMPLE: A waste paper basket is the shape of a frustum formed by removing the top 10 cm from a cone of height 50 cm and radius 35 cm. Find the volume of the waste paper basket to 3 significant figures.



Volume of original cone = $\frac{1}{3}\pi R^2 H = \frac{1}{3} \times \pi \times 35^2 \times 50 = 64140.850...\ cm^3$

Radius of removed cone = 35 ÷ 5 = 7 cm (because the cones are similar — the large cone is an enlargement of the small cone with scale factor 5)

Volume of removed cone = $\frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 7^2 \times 10 = 513.126...\ cm^3$

Volume of frustum = 64140.850... − 513.126... = 63627.723... = 63600 cm^3 (3 s.f.)

Rates of Flow

You need to be really careful with units in rates of flow questions. You might be given the dimensions of a shape in cm or m but the rate of flow in litres (e.g. litres per minute). Remember that 1 litre = 1000 cm^3.

EXAMPLE: A spherical fish tank with a radius of 15 cm is being filled with water at a rate of 4 litres per minute. How long will it take to fill the fish tank $\frac{2}{3}$ full (by volume)? Give your answer in minutes and seconds, to the nearest second.

Find the volume of the fish tank:

$V = \frac{4}{3}\pi r^3 = \frac{4}{3} \times \pi \times 15^3 = 14\ 137.166...\ cm^3$

So $\frac{2}{3}$ of the fish tank is: $\frac{2}{3} \times 14\ 137.166... = 9424.777...\ cm^3$

Then convert the rate of flow into cm^3/s:

4 litres per minute = 4000 cm^3/min = 66.666... cm^3/s

So it will take 9424.777... ÷ 66.666... = 141.371... seconds
= 2 minutes and 21 seconds (to the nearest second) to fill the fish tank.

THE BIG QUESTION

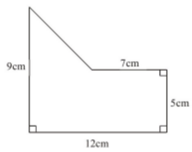


Diagram **NOT** accurately drawn

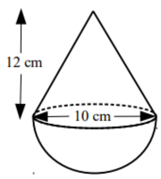
Work out the area of the shape.

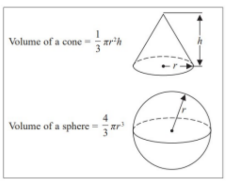
(Total 4 marks)

76 cm²

THE BIG QUESTION

The diagram shows a solid shape.
The shape is a cone on top of a hemisphere.





The height of the cone is 12 cm.
The base of the cone has a diameter of 10 cm.
The diameter of the hemisphere is 10 cm.

Work out the total volume of the solid shape.
Give your answer in terms of π.

(4 marks)

550π cm³

Simultaneous equations

What do I need to be able to do?

By the end of this unit you should be able to:

- Determine whether (x,y) is a solution
- Solve by substituting a known variable
- Solve by substituting an expression
- Solve graphically
- Solve by subtracting/ adding equations
- Solve by adjusting equations
- Form and solve linear simultaneous equations

Keywords

Solution: a value we can put in place of a variable that makes the equation true
Variable: a symbol for a number we don't know yet.
Equation: an equation says that two things are equal – it will have an equals sign =
Substitute: replace a variable with a numerical value
LCM: lowest common multiple (the first time the times table of two or more numbers match)
Eliminate: to remove
Expression: a maths sentence with a minimum of two numbers and at least one math operation (no equals sign)
Coordinate: a set of values that show an exact position
Intersection: the point two lines cross or meet

Is (x, y) a solution?

x and y represent values that can be substituted into an equation

Does the coordinate (1,8) lie on the line y=3x+5?

This coordinate represents x=1 and y=8

$y = 3x + 5$

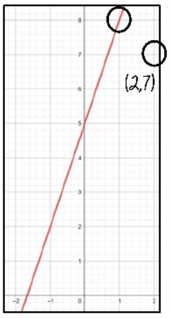
$8 = 3(1) + 5$

As the substitution makes the equation correct the coordinate (1,8) IS on the line y=3x+5

Is (2,7) on the same line?

$7 \neq 3(2) + 5$

No 7 does NOT equal 6+5



Substituting known variables

A line has the equation 3x + y = 14

Two different variables, two solutions

Stephanie knows the point x = 4 lies on that line. Find the value for y

$x = 4$

$3x + y = 14$ (x | x | x | y = 14)

$3(4) + y = 14$ (4 | 4 | 4 | y = 14)

$12 + y = 14$

$-12 \quad -12$

$y = 2$

Substituting in an expression

Substitute 2y in place of the x variable as they represent the same value

x = 2y (y | y = x)

x + y = 30 (x | y = 30)

Pair of simultaneous equations (two representations)

$x = 2y$ $x + y = 30$

(y | y | y = 30) $3y = 30$

$3y = 30$

$\div 3 \quad \div 3$

$y = 10$

$x = 2y$ (10 | 10 = x)

$x = 20$

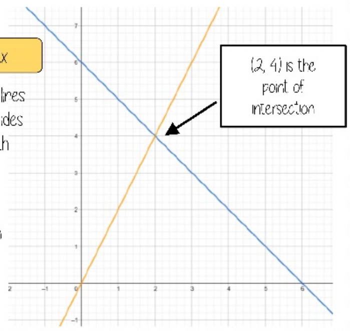
Solve graphically

x + y = 6   y = 2x

Linear equations are straight lines. The point of intersection provides the x and y solution for both equations

The solution that satisfies both equations is

$x = 2$ and $y = 4$



Solve by subtraction

$3x + 2y = 18$

$-\; x + 2y = 10$

$2x = 8$

$\div 2 \quad \div 2$

$x = 4$

$x + 2y = 10$

$(4) + 2y = 10$

$-4 \quad -4$

$2y = 6$

$\div 2 \quad \div 2$

$y = 3$

Bar model: (x x x y y = 18) − (x y y = 10) gives (x x = 8); $x = 4$, $y = 3$

Counters: x x x y y = 18; x y y = 10; → x x = 8; x = 4; y = 3

Solve by addition

Addition makes zero pairs

$3x + 2y = 16$

$+\; 6x - 2y = 2$

$9x = 18$

$\div 9 \quad \div 9$

$x = 2$

$3x + 2y = 16$

$3(2) + 2(y) = 16$

$6 + 2y = 16$

$-6 \quad -6$

$2y = 10$

$y = 5$

Counters: = 16; = 2; = 18; x = 2; y = 5

Solve by adjusting one

$h + j = 12$ No equivalent values

$2h + 2j = 29$

$2h + 2j = 24$

$2h + 2j = 29$

Bar models: 12, 29; 24, 29

By proportionally adjusting one of the equations – now solve the simultaneous equations choosing an addition or subtraction method

Solve by adjusting both

$2x + 3y = 39$

$5x - 2y = -7$

Counters: = 39; = −7

Use LCM to make equivalent x OR y values. Because of the negative values using zero pairs and y values is chosen choice

$4x + 6y = 78$

$15x - 6y = -21$

Counters: = 78; = −21

Now solve by addition

Addition makes zero pairs

2 Seven Steps for TRICKY Simultaneous Equations — H

EXAMPLE: Solve these two equations simultaneously: $7x + y = 1$ and $2x^2 - y = 3$

1. Rearrange the quadratic equation so that you have the non-quadratic unknown on its own. Label the two equations ① and ②.

$7x + y = 1$ — ① $y = 2x^2 - 3$ — ②

You could also rearrange the linear equation and substitute it into the quadratic.

2. Substitute the quadratic expression into the other equation. You'll get another equation — label it ③.

$7x + y = 1$ — ①
$y = 2x^2 - 3$ — ② ⇒ $7x + (2x^2 - 3) = 1$ — ③

Put the expression for y into equation ① in place of y.

3. Rearrange to get a quadratic equation. And guess what... You've got to solve it.

$2x^2 + 7x - 4 = 0$
$(2x - 1)(x + 4) = 0$
So $2x - 1 = 0$ OR $x + 4 = 0$
$x = 0.5$ OR $x = -4$

Remember — if it won't factorise, you can either use the formula or complete the square. Have a look at p.27-29 for more details.

4. Stick the first value back in one of the original equations (pick the easy one).

① $7x + y = 1$
Substitute in $x = 0.5$: $3.5 + y = 1$, so $y = 1 - 3.5 = -2.5$

5. Stick the second value back in the same original equation (the easy one again).

① $7x + y = 1$
Substitute in $x = -4$: $-28 + y = 1$, so $y = 1 + 28 = 29$

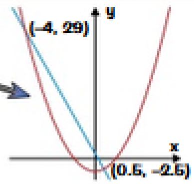
6. Substitute both pairs of answers back into the other original equation to check they work.

② $y = 2x^2 - 3$
Substitute in $x = 0.5$: $y = (2 \times 0.25) - 3 = -2.5$ — jolly good.
Substitute in $x = -4$: $y = (2 \times 16) - 3 = 29$ — smashing.

7. Write the pairs of answers out again, clearly, at the bottom of your working.

The two pairs of solutions are: $x = 0.5, y = -2.5$ and $x = -4, y = 29$

The solutions to simultaneous equations are actually the coordinates of the points where the graphs of the equations cross — so in this example, the graphs of $7x + y = 1$ and $2x^2 - y = 3$ will cross at (0.5, –2.5) and (–4, 29). There's more on this on p.52.



THE BIG QUESTION

Solve the simultaneous equations

$6x + 5y = 4.5$
$3x - 2y = 9$

$x = 2$
$y = -\frac{3}{2}$

THE BIG QUESTION

Solve the simultaneous equations

$x^2 + y^2 = 13$
$x = y - 5$

$x = -2$ or -3
$y = 3$ or 2

Straight Line Graphs

What do I need to be able to do?

By the end of this unit you should be able to:

- Compare gradients
- Compare intercepts
- Understand and use y= mx + c
- Find the equation of a line from a graph
- Interpret gradient and intercepts of real-life graphs

Keywords

Gradient: the steepness of a line

Intercept: where two lines cross. The y-intercept: where the line meets the y-axis.

Parallel: two lines that never meet with the same gradient.

Co-ordinate: a set of values that show an exact position on a graph.

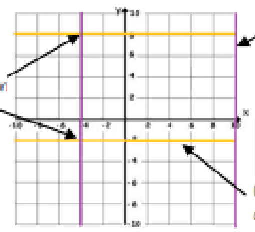
Linear: linear graphs (straight line) – linear common difference by addition/ subtraction

Asymptote: a straight line that a graph will never meet.

Reciprocal: a pair of numbers that multiply together to give 1

Perpendicular: two lines that meet at a right angle.

Lines parallel to the axes (R)



All the points on this line have a x coordinate of 10

'a' can be ANY positive or negative value including 0

Lines parallel to the **y axis** take the form **x = a** and are **vertical**

Lines parallel to the **x axis** take the form **y = a** and are **horizontal**

All the points on this line have a y coordinate of -2

e.g. (3, -2) (7, -2) (-2, -2) all lay on this line because the y coordinate is -2

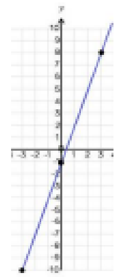
Plotting y = mx + c graphs (R)

$y = 3x - 1$ → 3 x the x coordinate then – 1

x	-3	0	3
y	-10	-1	8

Draw a table to display this information

This represents a coordinate pair (-3, -10)



You only need two points to form a straight line

Plotting more points helps you decide if your calculations are correct (if they do make a straight line)

Remember to join the points to make a line

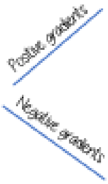
Compare Gradients

$y = mx + c$

The **coefficient** of x (the number in front of x) tells us the gradient of the line

The **greater** the gradient – the **steeper** the line

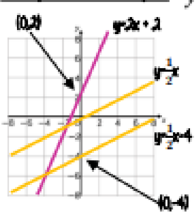
Parallel lines have the **same** gradient



Compare Intercepts

$y = mx + c$

The value of **c** is the point at which the line crosses the y-axis. **Y intercept**



The coordinate of a y intercept will always be (0,**c**)

Lines with the **same** y-intercept cross in the **same** place

y = mx + c

The **coefficient** of x (the number in front of x) tells us the gradient of the line

$y = mx + c$

The value of **c** is the point at which the line crosses the y-axis. **Y intercept**

y and x are **coordinates**

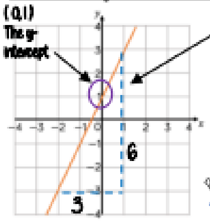
The equation of a line can be rearranged. E.g.

y = c + mx

c = y – mx

Identify which coefficient you are identifying or comparing

Find the equation from a graph



The Gradient $\frac{6}{3} = 2$

$y = 2x + 1$

The direction of the line indicates a positive gradient

Real life graphs

A plumber charges a £25 callout fee, and then £12.50 for every hour. Complete the table of values to show the cost of hiring the plumber.

Time (h)	0	1	2	3	8
Cost (£)	£25				£125

The y-intercept shows the minimum charge. The gradient represents the price per mile

In real life graphs like this values will always be positive because they measure distances or objects which cannot be negative.

Direct Proportion graphs To represent direct proportion the graph must start at the origin

When you have 0 pens this has 0 cost. The gradient shows the price per pen

A box of pens costs £2.30
Complete the table of values to show the cost of buying boxes of pens.

Boxes	0	1	2	3	8
Cost (£)		£2.30			

THE BIG QUESTION

A line passes through the point (0, 4).
The gradient of this line is 2.
Write down the equation of this line.

y = 2x + 4

Formulas and equations from Words

Formulas and Equations from Words

Making expressions or formulas from words can be a bit confusing as you're given a lot of information in one go. You just have to go through it slowly and carefully and extract the maths from it.

Make Expressions or Formulas from Given Information

Here are some of examples of how to use the information to write expressions and formulas.

EXAMPLE: Tiana is x years old. Leah is 5 years younger than Tiana. Abby is 4 times as old as Tiana. Find a simplified expression for the sum of their ages in terms of x.

Tiana's age is x
Leah's age is $x - 5$ — Leah is 5 years younger, so subtract 5
Abby's age is $4 \times x = 4x$ — 4 times older

The sum of their ages is: $x + (x - 5) + 4x = 6x - 5$

If you'd been told the sum of their ages, you'd have to set your expression equal to the sum and solve it to find x.

EXAMPLE: In rugby union, tries score 5 points and conversions score 2 points. A team scores a total of P points, made up of t tries and c conversions. Write a formula for P in terms of t and c.

Tries score 5 points — t tries will score $5 \times t = 5t$ points
Conversions score 2 points — c conversions will score $2 \times c = 2c$ points
So total points scored are $P = 5t + 2c$

Because you're asked for a formula, you must include the 'P = ' bit to get full marks (i.e. don't just put $5t + 2c$).

Use Your Expression to Solve Equations

Sometimes, you might be asked to use an expression to solve an equation.

EXAMPLE: A zoo has x zebras and four times as many lemurs. The difference between the number of zebras and the number of lemurs is 45. How many zebras does the zoo have?

The zoo has x zebras and $4 \times x = 4x$ lemurs.
The difference is $4x - x = 3x$, so $3x = 45$, which means $x = 15$. — Once you've formed the equation, you need to solve it to find the value of x.
So the zoo has 15 zebras.

EXAMPLE: Wendy, Naveed and Camilla give some books to charity. Naveed gives 6 more books than Wendy, and Camilla gives 7 more books than Naveed. Between them, they give away 46 books. How many books did they give each?

Let the number of books Wendy gives be x.
Then Naveed gives $x + 6$ books
and Camilla gives $(x + 6) + 7 = x + 13$ books
So in total they give $x + x + 6 + x + 13 = 3x + 19$ books

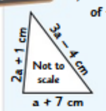
So $3x + 19 = 46$ — You're told this in the question.
$3x = 27$
$x = 9$
So Wendy gives 9 books, Naveed gives $9 + 6 = 15$ books and Camilla gives $15 + 7 = 22$ books.

Use Shape Properties to Find Formulas and Equations

In some questions, you'll need to use what you know about shapes (e.g. side lengths or areas) to come up with a formula or an equation to solve.

EXAMPLE:

a) Write a formula for P, the perimeter of the triangle below, in terms of a.



Form an expression for the perimeter:
$P = (a + 7) + (2a + 1) + (3a - 4)$
$P = 6a + 4$ cm

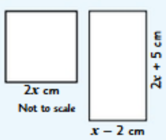
b) If the triangle has a perimeter of 58 cm, find the value of a.

$P = 58$, so set your formula equal to 58 and solve to find a:
$6a + 4 = 58$
$6a = 54$
$a = 9$

Compare Dimensions of Two Shapes to Find Equations

You might get a question that involves two shapes with related areas or perimeters — you'll have to use this fact to find side lengths or missing values.

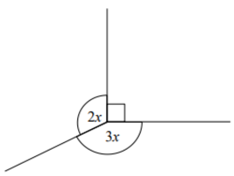
EXAMPLE: The perimeter of the rectangle is the same as the perimeter of the square. Find the value of x.



Perimeter of square $= 2x + 2x + 2x + 2x = 8x$ cm
Perimeter of rectangle $= (2x + 5) + (x - 2) + (2x + 5) + (x - 2)$
$= 6x + 6$ cm

Set the perimeter of the rectangle equal to the perimeter of the square and solve:
$8x = 6x + 6$
$2x = 6$
$x = 3$

THE BIG QUESTION



Find the value of x.

45 = x

Probability

What do I need to be able to do?

By the end of this unit you should be able to:

- Add, Subtract and multiply fractions
- Find probabilities using likely outcomes
- Use probability that sums to 1
- Estimate probabilities
- Use Venn diagrams and frequency trees
- Use sample space diagrams
- Calculate probability for independent events
- Use tree diagrams

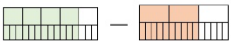
Keywords

Event: one or more outcomes from an experiment
Outcome: the result of an experiment
Intersection: elements (parts) that are common to both sets
Union: the combination of elements in two sets
Expected Value: the value/ outcome that a prediction would suggest you will get
Universal Set: the set that has all the elements
Systematic: ordering values or outcomes with a strategy and sequence
Product: the answer when two or more values are multiplied together

Add, Subtract and multiply fractions (R)

Addition and Subtraction

$\frac{4}{5} - \frac{2}{3}$



$\frac{12}{15} - \frac{10}{15} = \frac{2}{15}$

Use equivalent fractions to find a common multiple for both denominators

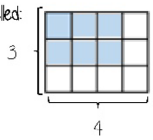
Multiplication

$\frac{3}{4} \times \frac{2}{3}$

$\frac{3}{4} \times \frac{2}{3} = \frac{6}{12}$

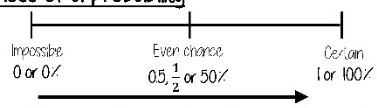
Parts shaded

Modelled:



Total number of parts in the diagram

Likeliness of a probability (R)



The more likely an event the further up the probability it will be in comparison to another event. (It will have a probability closer to 1)

Sum to 1 (R)

Probability is always a value between 0 and 1

The probability of getting a blue ball is $\frac{1}{5}$

∴The probability of NOT getting a blue ball is $\frac{4}{5}$

The sum of the probabilities is 1

Experimental data

Theoretical probability – What we expect to happen

Experimental probability – What actually happens when we try it out

The more trials that are completed the closer experimental probability and theoretical probability become

The probability becomes more accurate with more trials

Theoretical probability is proportional

Sample space (R)

The possible outcomes from rolling a dice

The possible outcomes from tossing a coin

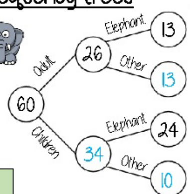
	1	2	3	4	5	6
H	1H	2H	3H	4H	5H	6H
T	1T	2T	3T	4T	5T	6T

P (Even number and tales) = $\frac{3}{12}$

Tables, Venn diagrams, Frequency trees

Frequency trees

60 people visited the zoo one Saturday morning. 26 of them were adults. 13 of the adult's favourite animal was an elephant. 24 of the children's favourite animal was an elephant.



Two-way table

	Adult	Child	Total
Elephant	13	24	37
Other	13	10	23
Total	26	34	60

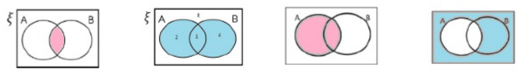
Frequency trees and two-way tables can show the same information

The total columns on two-way tables show the possible denominators

P(adult) = $\frac{26}{60}$

P(Child with favourite animal as elephant) = $\frac{13}{37}$

Venn diagram



in set A AND set B — $P(A \cap B)$

in set A OR set B — $P(A \cup B)$

in set A — $P(A)$

NOT in set A — $P(A')$

Independent events

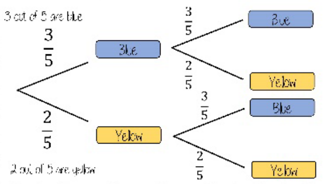
The outcome of two events happening. The outcome of the first event has no bearing on the outcome of the other

$P(A \text{ and } B) = P(A) \times P(B)$

Tree diagram for independent event

Isobel has a bag with 3 blue counters and 2 yellow. She picks a counter and replaces it before the second pick.

Because they are replaced the second pick has the same probability



$= P(BB) \quad \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$

$= P(BY) \quad \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$

$= P(YB) \quad \frac{2}{5} \times \frac{3}{5} = \frac{6}{25}$

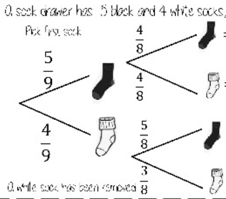
$= P(YY) \quad \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$

Dependent events

The outcome of the first event has an impact on the second event

Tree diagram for dependent event

A sock drawer has 5 black and 4 white socks, Jane picks 2 socks from the drawer.



$= P(BB) \quad \frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

$= P(BW) \quad \frac{5}{9} \times \frac{4}{8} = \frac{20}{72}$

$= P(WB) \quad \frac{4}{9} \times \frac{5}{8} = \frac{20}{72}$

$= P(WW) \quad \frac{4}{9} \times \frac{3}{8} = \frac{12}{72}$

NOTE: as "socks" are removed from the drawer the number of items in that drawer is also reduced ∴ the denominator is also reduced for the second pick.

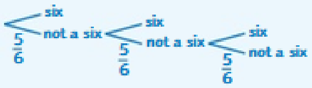
Look Out for 'At Least' Questions

When a question asks for '<u>at least</u>' a certain number of things happening, it's usually easier to work out (1 – probability of 'less than that number of things happening').

EXAMPLE: I roll 3 fair six-sided dice. Find the probability that I roll at least 1 six.

1) Rewrite this as 1 minus a probability. P(at least 1 six) = 1 – P(less than 1 six) = 1 – P(no sixes)

2) Work out P(no sixes). You can use a tree diagram — don't draw the whole thing, just the part you need.



P(no sixes) = $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$

So P(at least 1 six) = $1 - \frac{125}{216} = \frac{91}{216}$

The OR Rule gives P(At Least One Event Happens)

For two events, A and B... **P(A or B) = P(A) + P(B) – P(A and B)**

The probability of EITHER event A OR event B happening is equal to the two separate probabilities ADDED together MINUS the probability of events A AND B BOTH happening.

If the events A and B can't happen together then P(A and B) = 0 and the OR rule becomes:

When events can't happen together they're called mutually exclusive. **P(A or B) = P(A) + P(B)**

EXAMPLE: A spinner with red, blue, green and yellow sections is spun — the probability of it landing on each colour is shown in the table. Find the probability of spinning either red or green.

Colour	red	blue	yellow	green
Probability	0.25	0.3	0.35	0.1

The spinner can't land on both red and green so use the simpler OR rule. Just put in the probabilities.

P(red or green) = P(red) + P(green) = 0.25 + 0.1 = 0.35

THE BIG QUESTION

In a bag there are only red counters, blue counters, green counters and yellow counters.
A counter is taken at random from the bag.

The table shows the probabilities that the counter will be green or will be yellow.

Colour	Red	Blue	Green	Yellow
Probability			0.35	0.20

The probability that the counter will be red is twice the probability that the counter will be blue.

There are 21 green counters in the bag.

Work out the number of red counters in the bag.

Conditional Probability

Conditional probabilities crop up when you have dependent events — where one event affects another.

Using Conditional Probabilities — H

1) The conditional probability of A given B is the probability of event A happening given that event B happens.

You might see 'A given B' written as A|B.

2) Keep an eye out in questions for items being picked 'without replacement' — it's a tell-tale sign that it's going to be a conditional probability question.
3) If events A and B are independent then P(A given B) = P(A) and P(B given A) = P(B).

The AND rule for Conditional Probabilities

If events A and B are dependent (see p.110) then...

P(A and B) = P(A) × P(B given A)

The probability of events A AND B BOTH happening is equal to the probability of event A happening MULTIPLIED by the probability of event B happening GIVEN that event A happens.

EXAMPLE: Alia either watches TV or reads before bed. The probability she watches TV is 0.3. If she reads, the probability she is tired the next day is 0.8. What is the probability that Alia reads and isn't tired the next day?

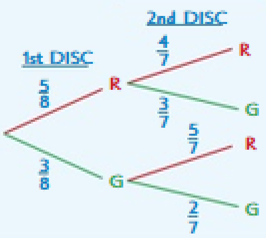
1) Label the events A and B.
   We want to find P(she reads AND isn't tired)
   So call "she reads" event A and "isn't tired" event B.
2) Use the information given in the question to work out the probabilities that you'll need to use the formula.
   P(A) = P(she reads) = 1 – 0.3 = 0.7
   P(B given A) = P(isn't tired given she reads) = 1 – 0.8 = 0.2
   P(A and B) = P(A) × P(B given A) = 0.7 × 0.2 = 0.14

Conditional Probabilities on Tree Diagrams

A good way to deal with conditional probability questions is to draw a tree diagram. The probabilities on a set of branches will change depending on the previous event.

This example was done 'with replacement' on p.111.

EXAMPLE: A box contains 5 red discs and 3 green discs. Two discs are taken at random without replacement. Find the probability that both discs are the same colour.



The probabilities for the 2nd pick depend on the colour of the 1st disc picked. This is because the 1st disc is not replaced.

P(both discs are red) = P(R and R) = $\frac{5}{8} \times \frac{4}{7} = \frac{20}{56}$

P(both discs are green) = P(G and G) = $\frac{3}{8} \times \frac{2}{7} = \frac{6}{56}$

P(both discs are same colour) = P(R and R or G and G) $= \frac{20}{56} + \frac{6}{56} = \frac{26}{56} = \frac{13}{28}$

THE BIG QUESTION

There are 10 counters in a bag.

5 of the counters are red.
3 of the counters are blue.
2 of the counters are green.

Billie takes two counters are taken at random from the bag.

Work out the probability that both of the counters Billie takes are the same colour.
You must show your working.

P(Red, Red) = $\frac{5}{10} \times \frac{4}{9} = \frac{20}{90}$

P(Blue, Blue) = $\frac{3}{10} \times \frac{2}{9} = \frac{6}{90}$

P(Green, Green) = $\frac{2}{10} \times \frac{1}{9} = \frac{2}{90}$

$\frac{20}{90} + \frac{6}{90} + \frac{2}{90} = \frac{28}{90}$

Sets and Venn diagrams

Venn diagrams are a way of displaying sets in intersecting circles — they're very pretty.

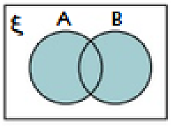
A Set is a Collection of Numbers or Objects

1) Sets are just collections of things — we call these 'things' elements.
2) Sets can be written in different ways but they'll always be in a pair of curly brackets {}. You can:
   - list the elements in the set, e.g. {2, 3, 5, 7}.
   - give a description of the elements in the set, e.g. {prime numbers less than 10}.
   - use formal notation, e.g. {x : x is a prime number less than 10}.

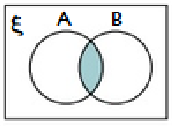
   Each of these describes the same set.
3) The symbol ∈ means 'is a member of'. So x ∈ A means 'x is a member of A'.
4) The universal set (ξ) is the group of things that the elements of the set are selected from.

Show Sets on Venn Diagrams

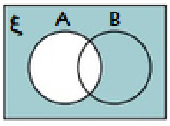
1) On a Venn diagram, each set is represented by a circle. The universal set is everything inside the rectangle.
2) The diagram can show either the actual elements of each set, or the number of elements in each set.



The union of sets A and B (written A ∪ B) contains all the elements in either set A or set B — it's everything inside the circles.



The intersection of sets A and B (written A ∩ B) contains all the elements in both set A and set B — it's where the circles overlap.

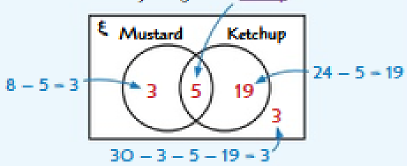


The complement of set A (written A') contains all members of the universal set that aren't in set A — it's everything outside circle A.

EXAMPLE: In a class of 30 pupils, 8 of them like mustard, 24 of them like ketchup and 5 of them like both mustard and ketchup.

a) Complete the Venn diagram below showing this information.

Start by filling in the overlap.



b) How many pupils like mustard or ketchup?

This is the number of pupils in the union of the two sets. 3 + 5 + 19 = 27

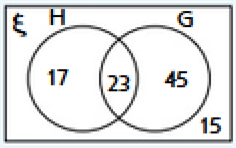
c) What is the probability that a randomly selected pupil will like mustard and ketchup?

5 out of 30 pupils are in the intersection. $\frac{5}{30} = \frac{1}{6}$

This is P(M ∩ K).

Finding Probabilities from Venn Diagrams H

EXAMPLE: The Venn diagram on the right shows the number of Year 10 pupils going on the History (H) and Geography (G) school trips.



Find the probability that a randomly selected Year 10 pupil is:

a) not going on the History trip.

n(Year 10 pupils) = 17 + 23 + 45 + 15 = 100

n(Not going on History trip) = 45 + 15 = 60

P(Not going on History trip) = $\frac{60}{100} = \frac{3}{5} = 0.6$

Use the formula from p.106 to find the probabilities.

b) not going on the History trip but going on the Geography trip.

n(Not going on History trip but going on Geography trip) = 45

P(Not going on History trip but going on Geography trip) = $\frac{45}{100} = \frac{9}{20} = 0.45$

c) going on the Geography trip given that they're not going on the History trip.

Careful here — think of this as selecting a pupil going on the Geography trip from those not going on the History trip.

P(Going on Geography trip given not going on History trip) = $\frac{45}{45 + 15} = \frac{45}{60} = \frac{3}{4} = 0.75$

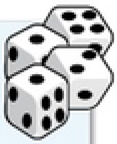
You could also use the conditional probability formula and your answers to parts a) and b).

Use the Product Rule to Count Outcomes

1) Sometimes it'll be difficult to list all the outcomes (e.g. if the number of outcomes is large or if there are more than two activities going on).
2) Luckily, you can count outcomes using the product rule.

The number of ways to carry out a combination of activities equals the number of ways to carry out each activity multiplied together.

EXAMPLE: Jason rolls four fair six-sided dice.



a) How many different ways are there to roll the four dice?

Each dice has 6 different ways that it can land (on 1, 2, 3, 4, 5 or 6).

Total number of ways of rolling four dice = 6 × 6 × 6 × 6 = 1296

b) How many different ways are there to only get even numbers when rolling the four dice?

Each dice has 3 different ways that it can land on an even number (on 2, 4, or 6).

Number of ways of only rolling even numbers = 3 × 3 × 3 × 3 = 81

c) What is the probability of only getting even numbers when rolling four dice?

$$P(\text{only even numbers}) = \frac{\text{number of ways to only get even numbers}}{\text{total number of ways to roll the dice}} = \frac{81}{1296} = \frac{1}{16}$$

THE BIG QUESTION

There are 20 people in a room.
Each person shakes each other person's hand once.

Work out the number handshakes that take place.

.... 190

Statistics

What do I need to be able to do?

By the end of this unit you should be able to:

- Construct and interpret frequency tables and polygon two-way tables, line, bar, & pie charts
- Find and interpret averages from a list and a table
- Construct and interpret time series graphs, stem and leaf diagrams and scatter graphs

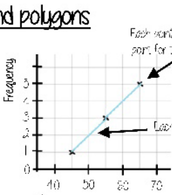
Keywords

Population: the whole group that is being studied
Sample: a selection taken from the population that will let you find out information about the larger group
Representative: a sample group that accurately represents the population
Random sample: a group completely chosen by chance. No predictability to who it will include
Bias: a built-in error that makes all values wrong by a certain amount
Primary data: data collected from an original source for a purpose
Secondary data: data taken from an external location. Not collected directly
Outlier: a value that stands apart from the data set

Frequency tables and polygons

x Weight(g)	Frequency
$40 < x \leq 50$	1
$50 < x \leq 60$	3
$60 < x \leq 70$	5

We do not know from grouped data where each value is placed so have to use an estimate for calculations



Each point is plotted at their mid point for the group it represents

Each point is connected with a straight line

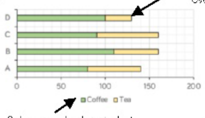
The data about weight starts at 40. So the axis can start at 40

MID POINTS
Mid-points are used as estimated values for grouped data. The mode of each group

Mid-point
$\frac{\text{Start point} + \text{End point}}{2}$

Bar and line charts

Composite bar charts

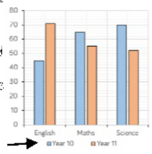


Compare the bars green compared to yellow. The size of each bar is the frequency.
Overall total easily comparable.

Categories clearly indicated

Dual bar charts

Bars are compared side by side
Easier to compare subgroups



Categories clearly indicated

Averages from a table (R)

Non-grouped data

Number of Siblings	0	1	2
Frequency	6	8	6
Subtotal	0	8	12

Overall Frequency 20

Total number of siblings 20

The data in a list: 0,0,0,0,0,0,1,1,1,1,1,1,1,1,2,2,2,2,2,2

Mean: $\frac{\text{total number of siblings}}{\text{Total frequency}}$ =

Grouped data

x Weight(g)	Frequency	Mid Point	MP x Freq
$40 < x \leq 50$	1	45	45
$50 < x \leq 60$	3	55	165
$60 < x \leq 70$	5	65	325

Overall Frequency: 9
Overall Total: 535
Mean: 62.8g

The data in a list: 45, 55, 55, 55, 65, 65, 65, 65, 65

Two way tables (R)

60 people visited the zoo one Saturday morning. 26 of them were adults. 13 of the adult's favourite animal was an elephant. 24 of the children's favourite animal was an elephant.

Extract information to input to the two-way table

Subgroups each need their own heading

	Adult	Child	Total
Elephant	13	24	37
Other	13	10	23
Total	26	34	60

Needs subgroup totals

Overall total

Draw and interpret Pie Charts (R)

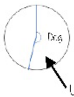
Type of pet	Dog	Cat	Hamster
Frequency	32	25	3

There were 60 people asked in this survey (Total frequency)

$\frac{32}{60}$ "32 out of 60 people had a dog"

This fraction of the 360 degrees represents dogs

$\frac{32}{60} \times 360 = 192°$



Use a protractor to draw. This is 192°

Multiplier method
as 60 goes into 360 – 6 times
Each frequency can be multiplied by 6 to find the degrees (proportion of 360)

Comparing Pie Charts
You NEED the overall frequency to make any comparisons

Averages from lists (R)

The Mean
a measure of average to find the central tendency… a typical value that represents the data

24, 8, 4, 11, 8,

Find the sum of the data (add the values)
55
Divide the overall total by how many pieces of data you have
55 ÷ 5

Mean = 11

The Mode (The modal value)
This is the number OR the item that occurs the most (it does not have to be numerical)

24, 8, 4, 11, 8,

This can still be easier if the data is ordered first

Mode = 8

The Median
The value in the center (in the middle) of the data

24, 8, 4, 11, 8,

Put the data in order 4, 8, 8, 11, 24
Find the value in the middle 4, 8, 8, 11, 24

Median = 8

NOTE: If there is no single middle value find the mean of the two numbers left.

For Grouped Data
The modal group – which group has the highest frequency

THE BIG QUESTION

Adam is measuring the heights in cm of his tomato plants.

Height (cm)	Frequency
140 < h ⩽ 150	7
150 < h ⩽ 160	10
160 < h ⩽ 170	15
170 < h ⩽ 180	19
180 < h ⩽ 200	9

(a) Estimate the mean height.

167.9 cm
(3)

What do I need to be able to do?

By the end of this unit you should be able to:

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Stem and leaf

A way to represent data and use to find averages

This stem and leaf diagram shows the age of people in a line at the supermarket.

0	7 9
1	4 5 6 8 8
2	1 3
3	0

Key: 1|4 Means 14 years old

Stem and leaf diagrams
Must include a key to explain what it represents
The information in the diagram should be ordered

Back to back stem and leaf diagrams

Girls		Boys
5	14	
7, 5, 5, 5, 4	15	3, 8, 9
8, 4, 2, 1, 0	16	2, 5, 7, 7, 7, 8, 8, 9
9, 8, 7, 6, 6, 4, 2, 1, 1, 0, 0	17	0, 2, 3, 6, 6, 7, 7
	18	0, 1, 4, 5

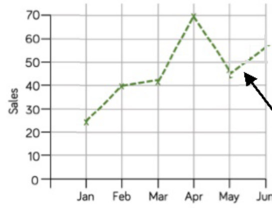
15 | 3.
Means 153 cm tall

Back to back stem and leaf diagrams
Allow comparisons of similar groups
Allow representations of two sets of data

Time-Series

R

This time-series graph shows the total number of car sales in £1000 over time



Look for general trends in the data. Some data shows a clear increase or a clear decrease over time.

Readings in-between points are estimates (on the dotted lines). You can use them to make assumptions.

Comparing distributions

Comparisons should include a statement of average and central tendency, as well as a statement about spread and consistency

Mean, mode, median – allows for a comparison about more or less average
Range – allows for a comparison about reliability and consistency of data

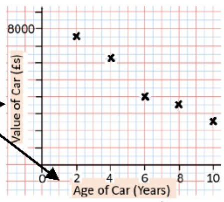
Draw and interpret a scatter graph.

R

Age of Car (Years)	2	4	6	8	10
Value of Car (£s)	7500	6250	4000	3500	2500

- This data may not be given in size order
- The data forms information pairs for the scatter graph
- Not all data has a relationship

All axes should be labelled



The axis should fit all the values on and be equally spread out

"This scatter graph show as the age of a car increases the value decreases"

The link between the data can be explained verbally

Linear Correlation

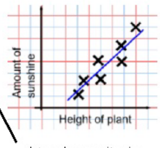
R

Positive Correlation	Negative Correlation	No Correlation
As one variable increases so does the other variable	As one variable increases the other variable decreases	There is no relationship between the two variables

The line of best fit

R

The Line of best fit is used to make estimates about the information in your scatter graph



Things to know:

- The line of best fit DOES NOT need to go through the origin (The point the axes cross)
- There should be approximately the same number of points above and below the line (It may not go through any points)
- The line extends across the whole graph

It is only an estimate because the line is designed to be an average representation of the data

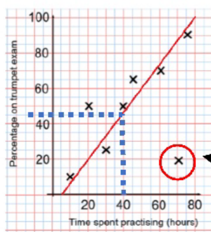
It is always a **straight line.**

Using a line of best fit

R

Interpolation is using the line of best fit to estimate values inside our data point.

eg 40 hours revising predicts a percentage of 45



Extrapolation is where we use our line of best fit to predict information outside of our data.
This is not always useful – in this example you cannot score more that 100%. So revising for longer can not be estimated

This point is an "outlier"
It is an outlier because it doesn't fit this model and stands apart from the data

THE BIG QUESTION

Here is a stem and leaf diagram showing the mass, in grams, of some apples.

15	6	6	7	9	
16	1	3	4	5	8
17	0	0	2	3	7
18	0	4	5		

Key: 15|6 = 156 grams

Work out the median mass.

....

Box Plots

The humble box plot might not look very fancy, but it gives you a useful summary of a data set.

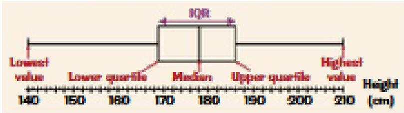
Box Plots show the Spread of a Data Set H

1) The lower quartile Q_1, the median Q_2 and the upper quartile Q_3 are the values 25% (¼), 50% (½) and 75% (¾) of the way through an ordered set of data.
So if a set of data has n values, you can work out the positions of the quartiles using these formulas:

Q_1: (n + 1)/4 Q_2: (n + 1)/2 Q_3: 3(n + 1)/4

2) The INTERQUARTILE RANGE (IQR) is the difference between the upper quartile and the lower quartile and contains the middle 50% of values.

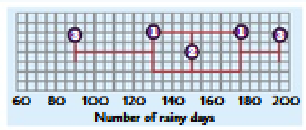
3) A box plot shows the minimum and maximum values in a data set and the values of the quartiles. But it doesn't tell you the individual data values.



EXAMPLE:

This table gives information about the numbers of rainy days last year in some cities.
On the grid below, draw a box plot to show the information.

Minimum number	90
Maximum number	195
Lower quartile	130
Median	150
Upper quartile	175

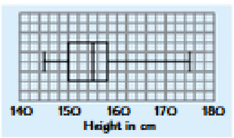


1. Mark on the quartiles and draw the box.
2. Draw a line at the median.
3. Mark on the minimum and maximum points and join them to the box with horizontal lines.

- Box plots show two measures of spread — range (highest – lowest) and interquartile range ($Q_3 - Q_1$).
- The range is based on all of the data values, so it can be affected by outliers — data values that don't fit the general pattern (i.e. that are a long way from the rest of the data).
- The IQR is based on only the middle 50% of the data values, so isn't affected by outliers. This means it can be a more reliable measure of spread than the range.

EXAMPLE:

This box plot shows a summary of the heights of a group of gymnasts.



a) Work out the range of the heights.
Range = highest – lowest = 175 – 145 = 30 cm

b) Work out the interquartile range for the heights.
Q_1 = 150 cm and Q_3 = 158 cm, so IQR = 158 – 150 = 8 cm

c) Do you think the range or the interquartile range is a more reliable measure of spread for this data? Give a reason for your answer.
The IQR is small and 75% of the values are less than 158 cm, so it's likely that the tallest height of 175 cm is an outlier. The IQR doesn't include the tallest height, so the IQR should be more reliable.

d) Explain whether or not it is possible to work out the number of gymnasts represented by the box plot.
The box plot gives no information about the number of values it represents, so it isn't possible to work out the number of gymnasts.

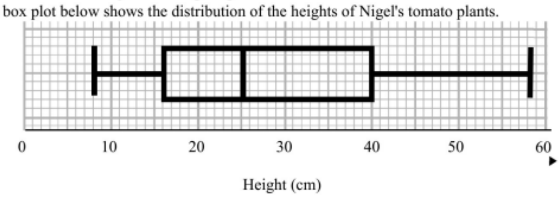
THE BIG QUESTION

The table shows some information about the heights, in cm, of tomato plants in Maggie's garden.

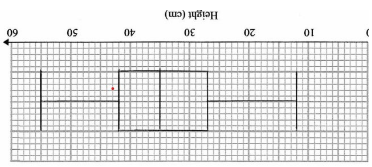
Minimum	Lower Quartile	Median	Upper Quartile	Maximum
12	27	35	42	55

(a) Sketch and label a box plot for this information.

There are also tomato plants in Nigel's garden.
The box plot below shows the distribution of the heights of Nigel's tomato plants.



(b) Compare the distribution of the heights of Maggie's plants with the distribution of height of Nigel's plants.



The median height of Maggie's tomatoes is greater
on average they are taller
The interquartile range of Maggie's tomatoes is less
they are less spread out.

Cumulative frequency

Cumulative frequency just means adding it up as you go along — i.e. the total frequency so far. You need to be able to draw a cumulative frequency graph and make estimates from it.

H

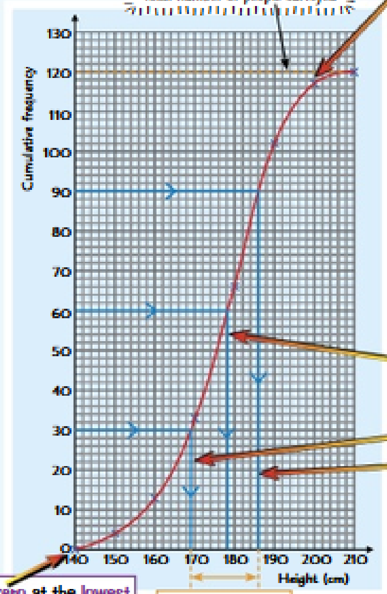
EXAMPLE: The table below shows information about the heights of a group of people.

a) Draw a cumulative frequency graph for the data.

b) Use your graph to estimate the median and interquartile range of the heights.

Height (h cm)	Frequency	Cumulative Frequency
140 < h ≤ 150	4	4
150 < h ≤ 160	9	4 + 9 = 13
160 < h ≤ 170	20	13 + 20 = 33
170 < h ≤ 180	33	33 + 33 = 66
180 < h ≤ 190	36	66 + 36 = 102
190 < h ≤ 200	15	102 + 15 = 117
200 < h ≤ 210	3	117 + 3 = 120

Total number of people surveyed



Plot zero at the lowest value in the first class.

Interquartile range

The values you read from the graph are estimates because they're based on grouped data — you don't know how the actual data values are spread within each class.

To Draw the Graph...

1) Add a 'CUMULATIVE FREQUENCY' COLUMN to the table — and fill it in with the RUNNING TOTAL of the frequency column.
2) PLOT points using the HIGHEST VALUE in each class and the CUMULATIVE FREQUENCY. (150, 4), (160, 13), etc.
3) Join the points with a smooth curve or straight lines.

To Find the Vital Statistics...

1) MEDIAN — go halfway up the side, across to the curve, then down and read off the bottom scale.
2) LOWER AND UPPER QUARTILES — go ¼ and ¾ up the side, across to the curve, then down and read off the bottom scale.
3) INTERQUARTILE RANGE — the distance between the lower and upper quartiles.

1) The halfway point is at ½ × 120 = 60. Reading across and down gives a median of 178 cm.

2) ¼ of the way up is at ¼ × 120 = 30. Reading across and down gives a lower quartile of 169 cm. ¾ of the way up is at ¾ × 120 = 90. Reading across and down gives an upper quartile of 186 cm.

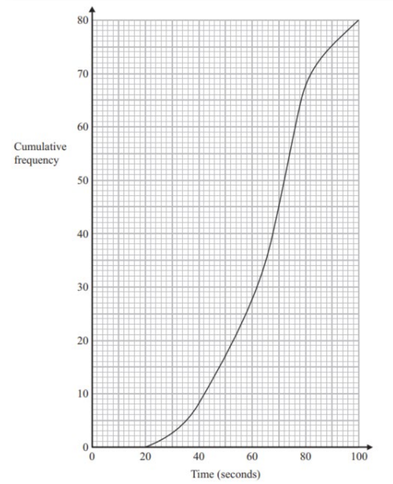
3) The interquartile range = 186 – 169 = 17 cm.

More Estimating...

To use the graph to estimate the number of values that are less than or greater than a given value:

Go along the bottom scale to the given value, up to the curve, then across to the cumulative frequency. (See the question below for an example.)

THE BIG QUESTION



Find the median time.

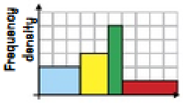
89 seconds

Histograms

A histogram is just a bar chart where the bars can be of different widths. This changes them from nice, easy-to-understand diagrams into seemingly incomprehensible monsters.

Histograms Show Frequency Density **H**

1) The vertical axis on a histogram is always called frequency density. You work it out using this formula:



Frequency Density = Frequency ÷ Class Width

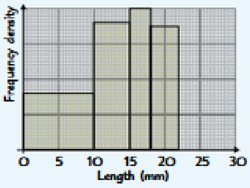
Remember... 'frequency' is just another way of saying 'how much' or 'how many'.

2) You can rearrange it to work out how much a bar represents.

Frequency = Frequency Density × Class Width = AREA of bar

EXAMPLE: This table and histogram show the lengths of beetles found in a garden.

Length (mm)	Frequency
$0 < x \le 10$	32
$10 < x \le 15$	36
$15 < x \le 18$	
$18 < x \le 22$	28
$22 < x \le 30$	16

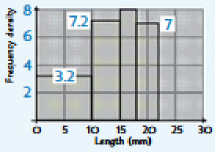


a) Use the histogram to find the missing entry in the table.

1) Add a frequency density column to the table and fill in what you can using the formula.

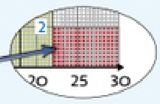
Frequency density
32 ÷ 10 = 3.2
36 ÷ 5 = 7.2
28 ÷ 4 = 7
16 ÷ 8 = 2

2) Use the frequency densities to label the vertical axis of the graph.



3) Now use the 3rd bar to find the frequency for the class "$15 < x \le 18$".
Frequency density = 8 and class width = 3.
So frequency = frequency density × class width = 8 × 3 = 24

b) Use the table to add the bar for the class "$22 < x \le 30$" to the histogram.
Frequency density = Frequency ÷ Class Width = $\frac{16}{8}$ = 2



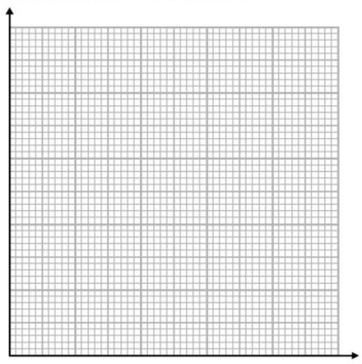
c) Estimate the number of beetles between 7.5 mm and 12.5 mm in length.
Use the formula frequency = frequency density × class width — multiply the frequency density of the class by the width of the part of that class you're interested in.

3.2 × (10 − 7.5) + 7.2 × (12.5 − 10)
= 3.2 × 2.5 + 7.2 × 2.5
= 26

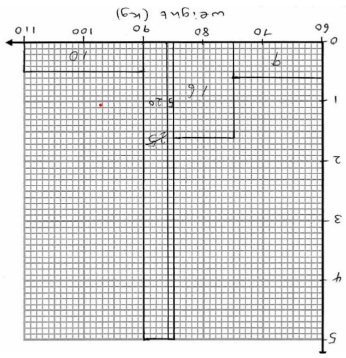
The table shows information about the weight of 60 pigs.

Weight (kg)	Frequency
60 < w ⩽ 75	9
75 < w ⩽ 85	16
85 < w ⩽ 90	25
90 < w ⩽ 110	10

(a) On the grid, draw a histogram for the information in the table.



(b) Find an estimate for the median.



.... 98

Comparing data

You can compare data sets using averages and range, or by drawing suitable diagrams.

Compare Data Sets Using Averages and Range

Say which data set has the higher/lower value and what that means in the context of the data.

EXAMPLE: Some children take part in a 'guess the weight of the baby hippo' competition. Here is some information about the weights they guess.

Compare the distributions of the weights guessed by the boys and the girls.

Boys:	Girls:
Mean = 40 kg	Mean = 34 kg
Median = 43 kg	Median = 33 kg
Range = 42 kg	Range = 30 kg

1. Compare averages:
The boys' mean and median values are higher than the girls', so the boys generally guessed heavier weights.

2. Compare ranges:
The boys' guesses have a bigger range, so the weights guessed by the boys show more variation.

You need to be able to compare the distributions of two sets of data represented by graphs and charts. That might mean comparing the shapes of the graphs, or reading off measures of average (mean, median or mode), and spread (range or interquartile range).

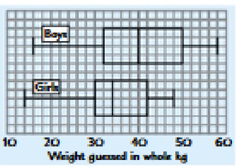
Compare Data Sets using Box Plots

H

For a reminder about box plots, see p.119.

From a box plot you can easily read off the median and work out the range and IQR.
Remember to say what these values mean in the context of the data.
A larger spread means the values are less consistent (there is more variation in the data).

EXAMPLE: An animal park is holding a 'guess the weight of the baby hippo' competition. These box plots summarise the weights guessed by a group of school children.



a) Compare the distributions of the weights guessed by the boys and the girls.

1) Compare averages by looking at the median values.
The median for the boys is higher than the median for the girls.
So the boys generally guessed heavier weights.

2) Compare the spreads by working out the range and IQR.
Boys' range = 58 – 16 = 42 and IQR = 50 – 32 = 18.
Girls' range = 48 – 14 = 34 and IQR = 42 – 30 = 12.
Both the range and the IQR are smaller for the girls' guesses, so there is less variation in the weights guessed by the girls.

It's important you give your answers in the context of the data.

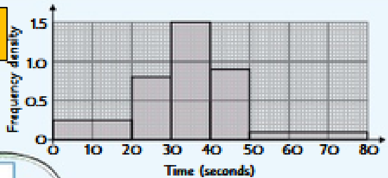
b) Can you tell from these box plots whether there are more boys or more girls in this group of children? Explain your answer.
The box plots don't show information on the numbers of data values, so you can't tell whether there are more boys or more girls.

Compare Data Sets using Histograms

See p.121 for a reminder about histograms.

EXAMPLE: This histogram shows information about the times taken by a large group of children to solve a puzzle.

H



a) Estimate the mean time taken to solve the puzzle.
Draw a table and fill in what the graph tells you.

Time (seconds)	Frequency Density	Frequency (f)	x	fx
0 < t ≤ 20	0.25	0.25 × 20 = 5	10	50
20 < t ≤ 30	0.8	0.8 × 10 = 8	25	200
30 < t ≤ 40	1.5	1.5 × 10 = 15	35	525
40 < t ≤ 50	0.9	0.9 × 10 = 9	45	405
50 < t ≤ 80	0.1	0.1 × 30 = 3	65	195
Total	—	40	—	1375

Find the frequency in each class using:
Frequency = Frequency Density × Class Width

Add a column for the mid-interval values.

Add up the 'Frequency × mid-interval value' column to estimate the total time taken.

Number of children

Mean = $\frac{\text{total time taken}}{\text{number of children}} = \frac{1375}{40}$ = 34.4 seconds (1 d.p.)

This is just like estimating the mean from a grouped frequency table (see p.118). Now you've found the frequencies, you could also find the class containing the median.

b) Write down the modal class.
Modal class is 30 < t ≤ 40

The modal class has the highest frequency density.
It's frequency density, not frequency, because the class widths vary.

c) Estimate the range of times taken.
Highest class boundary – lowest class boundary = 80 – 0 = 80 seconds

d) A large group of adults solve the same puzzle with a mean time of 27 seconds.
Is there any evidence to support the hypothesis that children take longer to solve the puzzle than adults?
Yes, there is evidence to support this hypothesis because the mean time for the children is longer.

Large samples mean the results should represent the population.

All Algebra content for foundation is pages 13 to 23

Brackets

factorising

Solving

Rearranging

This is Higher content

Iteration

Quadratic factorising

Completing a square

Algebraic fractions

Rearranging

Triple Brackets

1) For <u>three</u> brackets, just multiply <u>two</u> together as above, then multiply the result by the remaining bracket.

It doesn't matter which pair of brackets you multiply together first.

2) If you end up with <u>three terms</u> in one bracket, you <u>won't</u> be able to use FOIL. Instead, you can reduce it to a <u>series</u> of <u>single bracket multiplications</u> — like in the example below.

EXAMPLE: Expand and simplify $(x + 2)(x + 3)(2x - 1)$

$$(x + 2)(x + 3)(2x - 1) = (x + 2)(2x^2 + 5x - 3) = x(2x^2 + 5x - 3) + 2(2x^2 + 5x - 3)$$
$$= (2x^3 + 5x^2 - 3x) + (4x^2 + 10x - 6)$$
$$= 2x^3 + 9x^2 + 7x - 6$$

THE BIG QUESTION

Expand and Simplify $(3x + 1)(x + 2)(x - 4)$

$3x^3 - 5x^2 - 26x - 8$

Iteration

Iterative methods are techniques where you keep repeating a calculation in order to get closer and closer to the solution you want. You usually put the value you've just found back into the calculation to find a better value.

Where There's a Sign Change, There's a Solution

If you're trying to solve an equation that equals 0, there's one very important thing to remember:

> If there's a sign change (i.e. from positive to negative or vice versa) when you put two numbers into the equation, there's a solution between those numbers.

Think about the equation $x^3 - 3x - 1 = 0$. When $x = -1$, the expression gives $(-1)^3 - 3(-1) - 1 = 1$, which is positive, and when $x = -2$ the expression gives $(-2)^3 - 3(-2) - 1 = -3$, which is negative. This means that the expression will be 0 for some value between $x = -1$ and $x = -2$ (the solution).

Use Iteration When an Equation is Too Hard to Solve

Not all equations can be solved using the methods you've seen so far in this section (e.g. factorising, the quadratic formula etc.). But if you know an interval that contains a solution to an equation, you can use an iterative method to find the approximate value of the solution.

This is known as the decimal search method.

EXAMPLE: A solution to the equation $x^3 - 3x - 1 = 0$ lies between -1 and -2. By considering values in this interval, find a solution to this equation to 1 d.p.

1) Try (in order) the values of x with 1 d.p. that lie between −1 and −2. There's a sign change between −1.5 and −1.6, so the solution lies in this interval.
2) Now try values of x with 2 d.p. between −1.5 and −1.6. There's a sign change between −1.53 and −1.54, so the solution lies in this interval.
3) Both −1.53 and −1.54 round to −1.5 to 1 d.p. so a solution to $x^3 - 3x - 1 = 0$ is $x = -1.5$ to 1 d.p.

x	$x^3 - 3x - 1$	
−1.0	1	Positive
−1.1	0.969	Positive
−1.2	0.872	Positive
−1.3	0.703	Positive
−1.4	0.456	Positive
−1.5	0.125	Positive
−1.6	−0.296	Negative
−1.51	0.087049	Positive
−1.52	0.048192	Positive
−1.53	0.008423	Positive
−1.54	−0.032264	Negative

Each time you find a sign change, you narrow the interval that the solution lies within. Keep going until you know the solution to the accuracy you want.

EXAMPLE: Use the iteration machine below to find a solution to the equation $x^3 - 3x - 1 = 0$ to 1 d.p. Use the starting value $x_0 = -1$.

Look back at p.32 for more on the x_n notation.

1. Start with x_n → 2. Find the value of x_{n+1} by using the formula $x_{n+1} = \sqrt[3]{1 + 3x_n}$. → 3. If $x_n = x_{n+1}$ rounded to 1 d.p. then stop. If $x_n \neq x_{n+1}$ rounded to 1 d.p. go back to step 1 and repeat using x_{n+1}.

Put the value of x_0 into the iteration machine:

$x_0 = -1$ $\quad$ $x_1 = -1.25992... \neq x_0$ to 1 d.p.

$x_2 = -1.40605... \neq x_1$ to 1 d.p. $\quad$ $x_3 = -1.47639... \neq x_2$ to 1 d.p.

$x_4 = -1.50798... = x_3$ to 1 d.p.

x_3 and x_4 both round to −1.5 to 1 d.p. so the solution is $x = -1.5$ to 1 d.p.

This is the same example as above so the solution is the same.

THE BIG QUESTION

Using $\quad x_{n+1} = 3 + \dfrac{9}{{x_n}^2}$

With $x_0 = 3$

Find the values of x_1, x_2 and x_3.

$x_1 =$ 1.25

$x_2 =$ 1.095890411

$x_3 =$ 1.190199669

Factorising quadratics

So far so good. It gets a bit more complicated when 'a' isn't 1, but it's all good fun, right? Right? Well, I think it's fun anyway.

When 'a' is Not 1

The basic method is still the same but it's a bit messier — the initial brackets are different as the first terms in each bracket have to multiply to give 'a'. This means finding the other numbers to go in the brackets is harder as there are more combinations to try. The best way to get to grips with it is to have a look at an example.

EXAMPLE: Solve $3x^2 + 7x - 6 = 0$.

1) $3x^2 + 7x - 6 = 0$

2) $(3x \quad)(x \quad) = 0$

3) Number pairs: 1×6 and 2×3

(3x 1)(x 6) multiplies to give 18x and 1x which add/subtract to give 17x or 19x

(3x 6)(x 1) multiplies to give 3x and 6x which add/subtract to give 9x or 3x

(3x 3)(x 2) multiplies to give 6x and 3x which add/subtract to give 9x or 3x

(3x 2)(x 3) multiplies to give 9x and 2x which add/subtract to give 11x or 7x ✓

(3x 2)(x 3)

4) $(3x - 2)(x + 3)$

5) $(3x - 2)(x + 3) = 3x^2 + 9x - 2x - 6$
$= 3x^2 + 7x - 6$ ✓

6) $(3x - 2) = 0 \Rightarrow x = \frac{2}{3}$
$(x + 3) = 0 \Rightarrow x = -3$

1) Rearrange into the standard format.

2) Write down the initial brackets — this time, one of the brackets will have a 3x in it.

3) The tricky part: first, find pairs of numbers that multiply to give c (= 6), ignoring the minus sign for now.
Then, try out the number pairs you just found in the brackets until you find one that gives 7x. But remember, each pair of numbers has to be tried in 2 positions (as the brackets are different — one has 3x in it).

4) Now fill in the +/– signs so that 9 and 2 add/subtract to give +7 (= b).

5) ESSENTIAL check — EXPAND the brackets.

6) SOLVE THE EQUATION by setting each bracket equal to 0 (if a isn't 1, one of your answers will be a fraction).

EXAMPLE: Solve $2x^2 - 9x = 5$.

1) Put in standard form: $2x^2 - 9x - 5 = 0$

2) Initial brackets: $(2x \quad)(x \quad) = 0$

3) Number pairs: 1×5

(2x 5)(x 1) multiplies to give 2x and 5x which add/subtract to give 3x or 7x

(2x 1)(x 5) multiplies to give 1x and 10x which add/subtract to give 9x or 11x

(2x 1)(x 5) ✓

4) Put in the signs: $(2x + 1)(x - 5)$

5) Check:
$(2x + 1)(x - 5) = 2x^2 - 10x + x - 5$
$= 2x^2 - 9x - 5$ ✓

6) Solve:
$(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}$
$(x - 5) = 0 \Rightarrow x = 5$

THE BIG QUESTION

Factorise $3x^2 + 16x + 21$

(3x + 7)(x + 3)

The solutions to ANY quadratic equation $ax^2 + bx + c = 0$ are given by this formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

LEARN THIS FORMULA — and how to use it. Using it isn't that hard, but there are a few pitfalls — so TAKE HEED of these crucial details:

Quadratic Formula — Five Crucial Details

1) Take it nice and slowly — always write it down in stages as you go.
2) **WHENEVER YOU GET A MINUS SIGN, THE ALARM BELLS SHOULD ALWAYS RING!**
3) Remember it's '2a' on the bottom line, not just 'a' — and you divide ALL of the top line by 2a.
4) The ± sign means you end up with two solutions (by replacing it in the final step with '+' and '–').
5) If you get a negative number inside your square root, go back and check your working. Some quadratics do have a negative value in the square root, but they won't come up at GCSE.

If either 'a' or 'c' is negative, the -4ac effectively becomes +4ac, so watch out. Also, be careful if b is negative, as -b will be positive.

EXAMPLE: Solve $3x^2 + 7x = 1$, giving your answers to 2 decimal places.

$3x^2 + 7x - 1 = 0$

$a = 3, \quad b = 7, \quad c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7 \pm \sqrt{7^2 - 4 \times 3 \times -1}}{2 \times 3}$$

$$= \frac{-7 \pm \sqrt{49 + 12}}{6}$$

$$= \frac{-7 \pm \sqrt{61}}{6}$$

$$= \frac{-7 + \sqrt{61}}{6} \text{ or } \frac{-7 - \sqrt{61}}{6}$$

$= 0.1350\ldots$ or $-2.468\ldots$

So to 2 d.p. the solutions are: $x = 0.14$ or -2.47

1) First get it into the form $ax^2 + bx + c = 0$.
2) Then carefully identify a, b and c.
3) Put these values into the quadratic formula and write down each stage.
4) Finally, as a check put these values back into the original equation:
   E.g. for $x = 0.1350$: $3 \times 0.135^2 + 7 \times 0.135 = 0.999675$, which is 1, as near as...

Notice that you do two calculations at the final stage — one + and one –

THE BIG QUESTION

Solve $x^2 - 4x - 1 = 0$

Give your answers in the form $a \pm \sqrt{b}$.

$2 \pm \sqrt{5}$

Rearranging equations

Carrying straight on from the previous page, now it's time for what to do if...

...there's a Square or Square Root Involved

If the subject appears as a square or in a square root, you'll have to use steps 1 and 7 (not necessarily both).

EXAMPLE: Make u the subject of the formula $v^2 = u^2 + 2as$.

There aren't any square roots, fractions or brackets so ignore steps 1-3 (this is pretty easy so far).

4) Collect all the subject terms on one side and all non-subject terms on the other.

$(-2as) \quad u^2 = v^2 - 2as$

5) It's now in the form $Au^2 = B$ (where $A = 1$ and $B = v^2 - 2as$)

This is a real-life equation — v = final velocity, u = initial velocity, a = acceleration and s = displacement.

A = 1, which means it's already in the form '$u^2 =$ ', so ignore step 6.

7) Square root both sides to get '$u = \pm$ '.

$(\sqrt{\ }) \quad u = \pm\sqrt{v^2 - 2as}$

EXAMPLE: Make n the subject of the formula $2(m + 3) = \sqrt{n + 5}$.

1) Get rid of any square roots by squaring both sides.

$[2(m + 3)]^2 = (\sqrt{n + 5})^2$

$4(m^2 + 6m + 9) = n + 5$

$4m^2 + 24m + 36 = n + 5$

There aren't any fractions so ignore step 2.
The brackets were removed when squaring so ignore step 3.

4) Collect all the subject terms on one side and all non-subject terms on the other.

$(-5) \quad n = 4m^2 + 24m + 31$ This is in the form '$n =$ ' so you don't need to do steps 5-7.

...the Subject Appears Twice

Go home and cry. No, not really — you'll just have to do some factorising, usually in step 5.

EXAMPLE: Make p the subject of the formula $q = \frac{p + 1}{p - 1}$.

There aren't any square roots so ignore step 1.

2) Get rid of any fractions. $q(p - 1) = p + 1$

3) Multiply out any brackets. $pq - q = p + 1$

4) Collect all the subject terms on one side and all non-subject terms on the other.

$pq - p = q + 1$

5) Combine like terms on each side of the equation. $p(q - 1) = q + 1$

This is where you factorise — p was in both terms on the LHS so it comes out as a common factor.

6) Divide both sides by $(q - 1)$ to give '$p =$ '. $p = \frac{q + 1}{q - 1}$ (p isn't squared, so you don't need step 7.)

THE BIG QUESTION

Make x the subject of the formula $\frac{a}{b} = \frac{2x}{x+5}$

$\frac{5a}{2b - a} = x$

Completing a square

There's just one more method to learn for solving quadratics — and it's a bit of a nasty one. It's called 'completing the square', and takes a bit to get your head round it.

Solving Quadratics by 'Completing the Square'

To 'complete the square' you have to:

1) Write down a SQUARED bracket, and then
2) Stick a number on the end to 'COMPLETE' it.

$$x^2 + 12x - 5 = (x + 6)^2 - 41$$

The SQUARE... ...COMPLETED

It's not that bad if you learn all the steps — some of them aren't all that obvious.

1) As always, REARRANGE THE QUADRATIC INTO THE STANDARD FORMAT: $ax^2 + bx + c$ (the rest of this method is for $a = 1$).
2) WRITE OUT THE INITIAL BRACKET: $(x + \frac{b}{2})^2$ — just divide the value of b by 2.
3) MULTIPLY OUT THE BRACKETS and COMPARE TO THE ORIGINAL to find what you need to add or subtract to complete the square.
4) Add or subtract the ADJUSTING NUMBER to make it MATCH THE ORIGINAL.

If a isn't 1, you have to divide through by 'a' or take out a factor of 'a' at the start — see next page.

EXAMPLE:

a) Express $x^2 + 8x + 5$ in the form $(x + m)^2 + n$.

1) It's in the standard format. — $x^2 + 8x + 5$
2) Write out the initial bracket — $(x + 4)^2$
3) Multiply out the brackets and compare to the original. — $(x + 4)^2 = x^2 + 8x + 16$ (Original equation had +5 here...)
4) Subtract adjusting number (11). — $(x + 4)^2 - 11 = x^2 + 8x + 16 - 11$ (...so you need −11) $= x^2 + 8x + 5$ ✓ (matches original now!)

So the completed square is: $(x + 4)^2 - 11$.

b) Hence solve $x^2 + 8x + 5 = 0$, leaving your answers in surd form.

Now use the completed square to solve the equation. There are three more steps for this:

$(x + 4)^2 - 11 = 0$

1) Put the number on the other side (+11). — $(x + 4)^2 = 11$
2) Square root both sides (don't forget the ±!) ($\sqrt{\ }$). — $x + 4 = \pm\sqrt{11}$
3) Get x on its own (−4). — $x = -4 \pm \sqrt{11}$

So the two solutions (in surd form) are: $x = -4 + \sqrt{11}$ and $x = -4 - \sqrt{11}$

If you really don't like steps 3-4, just remember that the value you need to add or subtract is always $c - \left(\frac{b}{2}\right)^2$.

THE BIG QUESTION

(a) Write $x^2 - 6x + 1$ in the form $(x + a)^2 + b$ where a and b are integers.

(b) Hence, or otherwise, write down the coordinates of the turning point of the graph of $y = x^2 - 6x + 1$

$(x - 3)^2 - 8$

(2)

(3, −8)

If you're a fan of completing the square, good news — there's another page on it here.
If you're not a fan of completing the square, bad news — there's another page on it here.

Completing the Square When 'a' Isn't 1

If 'a' isn't 1, completing the square is a bit trickier. You follow the same method as on the previous page, but you have to take out a factor of 'a' from the x^2 and x-terms before you start (which often means you end up with awkward fractions). This time, the number in the brackets is $\frac{b}{2a}$.

EXAMPLE: Write $2x^2 + 5x + 9$ in the form $a(x + m)^2 + n$.

1) It's in the standard format. — $2x^2 + 5x + 9$
2) Take out a factor of 2. — $2(x^2 + \frac{5}{2}x) + 9$
3) Write out the initial bracket. — $2(x + \frac{5}{4})^2$
4) Multiply out the bracket and compare to the original. — $2(x + \frac{5}{4})^2 = 2x^2 + 5x + \frac{25}{8}$
5) Add on adjusting number ($\frac{47}{8}$). — $2(x + \frac{5}{4})^2 + \frac{47}{8} = 2x^2 + 5x + \frac{25}{8} + \frac{47}{8}$
$= 2x^2 + 5x + 9$ ✓ — matches original now!

Original equation had +9 here...
...so you need $9 - \frac{25}{8} = \frac{47}{8}$

So the completed square is: $2(x + \frac{5}{4})^2 + \frac{47}{8}$

The Completed Square Helps You Sketch the Graph

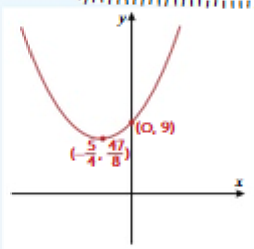
There's more about sketching quadratic graphs on p.48, but you can use the completed square to work out key details about the graph — like the turning point (maximum or minimum) and whether it crosses the x-axis.

1) For a positive quadratic (where the x^2 coefficient is positive), the adjusting number tells you the minimum y-value of the graph. If the completed square is $a(x + m)^2 + n$, this minimum y-value will occur when the brackets are equal to 0 (because the bit in brackets is squared, so is never negative) — i.e. when $x = -m$.
2) The solutions to the equation tell you where the graph crosses the x-axis. If the adjusting number is positive, the graph will never cross the x-axis as it will always be greater than 0 (this means that the quadratic has no real roots).

EXAMPLE: Sketch the graph of $y = 2x^2 + 5x + 9$.

From above, completed square form is $2(x + \frac{5}{4})^2 + \frac{47}{8}$.
The minimum point occurs when the brackets are equal to 0 — this will happen when $x = -\frac{5}{4}$.
At this point, the graph takes its minimum value, which is the adjusting number ($\frac{47}{8}$).
The adjusting number is positive, so the graph will never cross the x-axis.
Find where the curve crosses the y-axis by substituting $x = 0$ into the equation and mark this on your graph. $y = 0 + 0 + 9 = 9$

This is only a sketch, so label the points you know



Algebraic Fractions

Unfortunately, fractions aren't limited to numbers — you can get algebraic fractions too.
Fortunately, everything you learnt about fractions on p.5-6 can be applied to algebraic fractions as well.

Simplifying Algebraic Fractions

You can simplify algebraic fractions by cancelling terms on the top and bottom — just deal with each letter individually and cancel as much as you can. You might have to factorise first (see pages 19 and 25-26).

EXAMPLES:

1. Simplify $\frac{21x^3y^2}{14xy^3}$

$\frac{21x^3y^2}{14xy^3} = \frac{3x^2}{2y}$

÷7 on the top and bottom
÷x on the top and bottom to leave x^2 on the top
÷y^2 on the top and bottom to leave y on the bottom

2. Simplify $\frac{x^2-16}{x^2+2x-8}$

Factorise the top using D.O.T.S.
Factorise the quadratic on the bottom

$\frac{(x+4)(x-4)}{(x-2)(x+4)} = \frac{x-4}{x-2}$

Then cancel the common factor of $(x+4)$

Multiplying/Dividing Algebraic Fractions

1) To multiply two fractions, just multiply tops and bottoms separately.
2) To divide, turn the second fraction upside down then multiply.

EXAMPLE: Simplify $\frac{x^2-4}{x^2+x-12} \div \frac{2x+4}{x^2-3x}$

Turn the second fraction upside down — Factorise and cancel — Multiply tops and bottoms

$$\frac{x^2-4}{x^2+x-12} \div \frac{2x+4}{x^2-3x} = \frac{x^2-4}{x^2+x-12} \times \frac{x^2-3x}{2x+4} = \frac{(x+2)(x-2)}{(x+4)(x-3)} \times \frac{x(x-3)}{2(x+2)} = \frac{x-2}{x+4} \times \frac{x}{2} = \frac{x(x-2)}{2(x+4)}$$

Adding/Subtracting Algebraic Fractions

Adding or subtracting is a bit more difficult:
1) Work out the common denominator (see p.6).
2) Multiply top and bottom of each fraction by whatever gives you the common denominator.
3) Add or subtract the numerators only.

For the common denominator, find something both denominators divide into.

Fractions		
$\frac{1}{x} + \frac{1}{3x}$	$\frac{1}{x+1} + \frac{1}{x-2}$	$\frac{1}{x} + \frac{1}{x(x+1)}$
$3x$	$(x+1)(x-2)$	$x(x+1)$
Common denominator		

EXAMPLE: Write $\frac{3}{(x+3)} + \frac{1}{(x-2)}$ as a single fraction.

Common denominator will be $(x+3)(x-2)$
1st fraction: × top & bottom by $(x-2)$
2nd fraction: × top & bottom by $(x+3)$
Add the numerators

$$\frac{3}{(x+3)} + \frac{1}{(x-2)} = \frac{3(x-2)}{(x+3)(x-2)} + \frac{(x+3)}{(x+3)(x-2)} = \frac{3x-6}{(x+3)(x-2)} + \frac{x+3}{(x+3)(x-2)} = \frac{4x-3}{(x+3)(x-2)}$$

THE BIG QUESTION

Simplify fully $\frac{x^2+5x}{x^2+7x+10}$

Solve $\frac{8}{x+3} + \frac{3}{x+8} = 1$

$\frac{x}{x+2}$

$x = -7$ or 7

Year 11 Knowledge Organiser

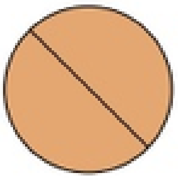
Contents pages

Circles and Sectors

Yes, I thought I could detect some groaning when you realised that this is another page of formulas. You know the drill...

LEARN these Formulas

Area and Circumference of Circles



Area of circle = $\pi \times (\text{radius})^2$
Remember that the radius is half the diameter.

$A = \pi r^2$

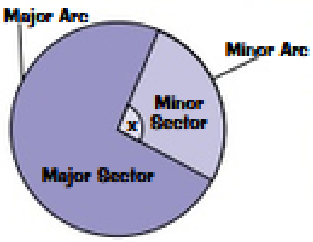
For these formulas, use the π button on your calculator. For non-calculator questions, use $\pi = 3.142$.

Circumference = $\pi \times$ diameter
= $2 \times \pi \times$ radius

$C = \pi D = 2\pi r$

Areas of Sectors and Segments H

These next ones are a bit more tricky — before you try and learn the formulas, make sure you know what a sector, an arc and a segment are (I've helpfully labelled the diagrams below — I'm nice like that).

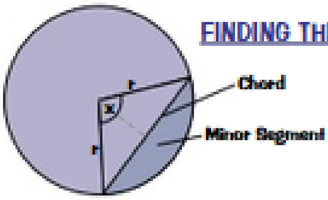


$$\text{Area of Sector} = \frac{x}{360} \times \text{Area of full Circle}$$

(Pretty obvious really, isn't it?)

$$\text{Length of Arc} = \frac{x}{360} \times \text{Circumference of full Circle}$$

(Obvious again, no?)



FINDING THE AREA OF A SEGMENT is OK if you know the formulas.

1) Find the area of the sector using the above formula.
2) Find the area of the triangle, then subtract it from the sector's area. You can do this using the '½ ab sin C' formula for the area of the triangle (see previous page), which becomes: $\frac{1}{2} r^2 \sin x$.

EXAMPLE: In the diagram on the right, a sector with angle 60° has been cut out of a circle with radius 3 cm. Find the exact area of the shaded shape.

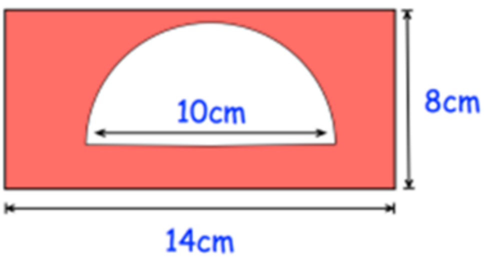
First find the angle of the shaded sector (this is the major sector):
360° – 60° = 300°

Then use the formula to find the area of the shaded sector:

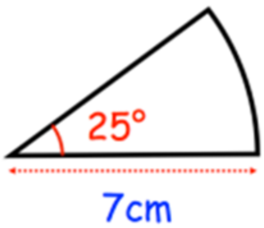
$$\text{area of sector} = \frac{x}{360} \times \pi r^2 = \frac{300}{360} \times \pi \times 3^2$$
$$= \frac{5}{6} \times \pi \times 9 = \frac{15}{2}\pi \text{ cm}^2$$

'Exact area' means leave your answer in terms of π.

Calculate the shaded area



Find the area of the sector



THE BIG QUESTION

72.7cm²
10.7cm²

Angle Rules

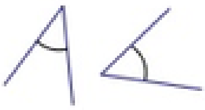
Before we really get going with the thrills and chills of angles and geometry, there are a few things you need to know. Nothing too scary — just some special angles and some fancy notation.

Fancy Angle Names

Some angles have special names. You might have to identify these angles in the exam.

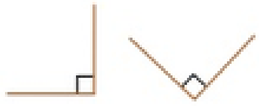
ACUTE angles

Sharp pointy ones (less than 90°)



RIGHT angles

Square corners (exactly 90°)

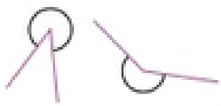


OBTUSE angles

Flatter ones (between 90° and 180°)

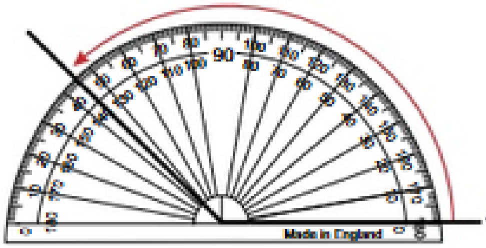
REFLEX angles

Ones that bend back on themselves (more than 180°)



Measuring Angles with a Protractor

1) ALWAYS position the protractor with the base line of it along one of the lines as shown here:



2) Count the angle in 10° STEPS from the start line right round to the other line over there.

Start line

Check your measurement by looking at it. If it's between a right angle and a straight line, it's between 90° and 180°.

DON'T JUST READ A NUMBER OFF THE SCALE — chances are it'll be the wrong one because there are TWO scales to choose from.

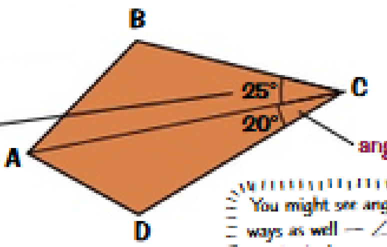
The answer here is 135° (NOT 45°) which you will only get right if you start counting 10°, 20°, 30°, 40° etc. from the start line until you reach the other line.

Three-Letter Angle Notation

The best way to say which angle you're talking about in a diagram is by using THREE letters.

For example in the diagram, angle ACB = 25°.

1) The middle letter is where the angle is.
2) The other two letters tell you which two lines enclose the angle.



angle ACD = 20°

You might see angles written in other ways as well — ∠ABC and $\hat{ABC}$ are both the same as angle ABC.

If you were an angle you'd be acute one...

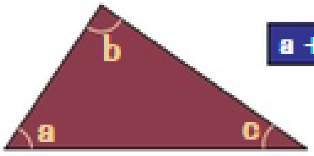
If an exam question asks you to write down the 'special name' for a particular angle, don't put 'honeybunch' — they want one of the fancy names above. Learn the page then have a bash at this Exam Practice Question.

Q1 a) An angle measures 66°. What type of angle is this? [1 mark]
b) Measure ∠ADC on the diagram above. [1 mark]

If you know **all** these rules **thoroughly**, you'll at least have a fighting chance of working out problems with lines and angles. If you don't — you've no chance. Sorry to break it to you like that.

5 Simple Rules — that's all

1) Angles in a triangle add up to 180°.

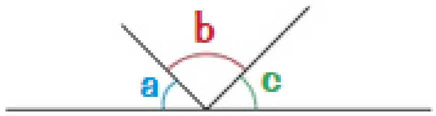


$a + b + c = 180°$

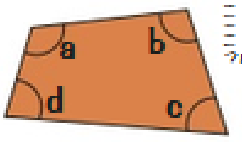
There's a nice proof of this (using parallel lines) on the next page.

2) Angles on a straight line add up to 180°.

$a + b + c = 180°$



3) Angles in a quadrilateral add up to 360°.

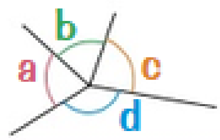


Remember that a quadrilateral is a 4-sided shape.

$a + b + c + d = 360°$

You can see why this is if you split the quadrilateral into two triangles along a diagonal. Each triangle has angles adding up to 180°, so the two together have angles adding up to 180° + 180° = 360°.

4) Angles round a point add up to 360°.



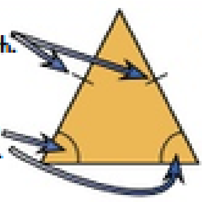
$a + b + c + d = 360°$

5) Isosceles triangles have 2 sides the same and 2 angles the same.

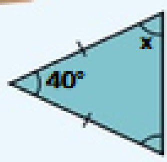
In an isosceles triangle, you only need to know one angle to be able to find the other two.

These dashes indicate two sides the same length.

These angles are the same.



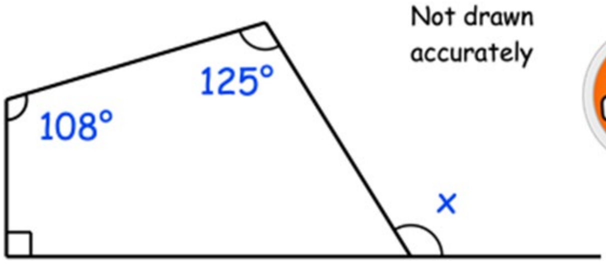
EXAMPLE: Find the size of angle x.



180° − 40° = 140°

The two angles on the right are the same (they're both x) and they must add up to 140°, so $2x = 140°$, which means $x = 70°$.

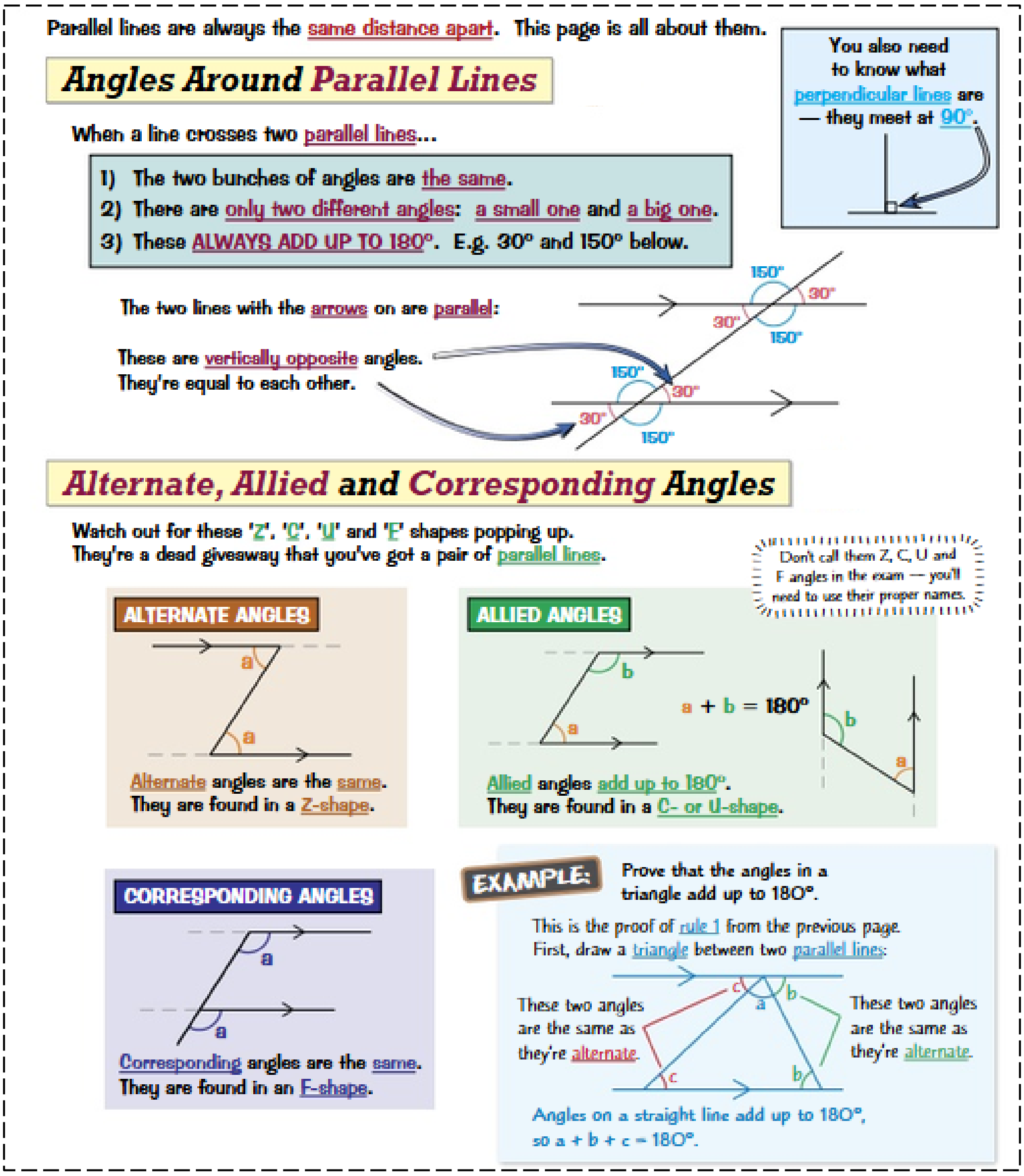
Not drawn accurately



THE BIG QUESTION

Work out the size of the angle marked x.

Angles in Parallel Lines



Parallel lines are always the same distance apart. This page is all about them.

Angles Around Parallel Lines

When a line crosses two parallel lines...

1) The two bunches of angles are the same.
2) There are only two different angles: a small one and a big one.
3) These ALWAYS ADD UP TO 180°. E.g. 30° and 150° below.

You also need to know what perpendicular lines are — they meet at 90°.

The two lines with the arrows on are parallel:

These are vertically opposite angles. They're equal to each other.

Alternate, Allied and Corresponding Angles

Watch out for these 'Z', 'C', 'U' and 'F' shapes popping up. They're a dead giveaway that you've got a pair of parallel lines.

Don't call them Z, C, U and F angles in the exam — you'll need to use their proper names.

ALTERNATE ANGLES

Alternate angles are the same. They are found in a Z-shape.

ALLIED ANGLES

a + b = 180°

Allied angles add up to 180°. They are found in a C- or U-shape.

CORRESPONDING ANGLES

Corresponding angles are the same. They are found in an F-shape.

EXAMPLE: Prove that the angles in a triangle add up to 180°.

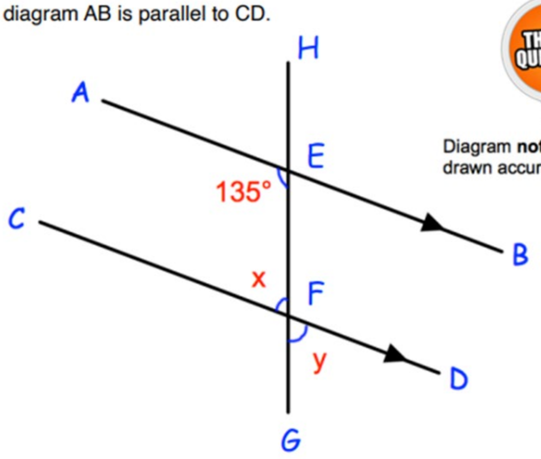
This is the proof of rule 1 from the previous page. First, draw a triangle between two parallel lines:

These two angles are the same as they're alternate.

These two angles are the same as they're alternate.

Angles on a straight line add up to 180°, so a + b + c = 180°.

In the diagram AB is parallel to CD.



THE BIG QUESTION

Diagram **not** drawn accurately

(a) Work out the size of the angle marked x. Give a reason for your answer.

(b) Write down the value of y. Give a reason for your answer.

x = 45° allied (co-interior) angles sum to 180°

y = 45° vertically opposite angles are equal

Geometry Problems

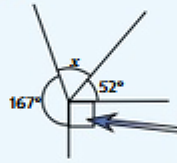
As if geometry wasn't enough of a problem already, here's a page dedicated to geometry problems. Make sure you learn the five angle rules on p.88 — they'll help a lot on these questions. Pinky promise.

Using the Five Angle Rules

The best method is to find whatever angles you can until you can work out the ones you're looking for. It's a bit trickier when you have to use more than one rule, but writing them all down is a big help.

EXAMPLES:

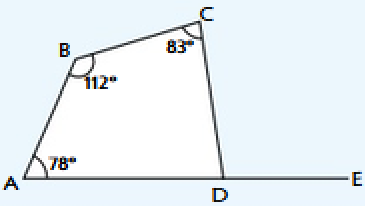
1. Find the value of x.



Remember — this little square means that it's a right angle (90°).

Use rule 4 from p.88:
Angles round a point add up to 360°,
so $x + 52° + 90° + 167° = 360°$
$x = 360° - 52° - 90° - 167° = 51°$

2. Find the size of angle CDE.



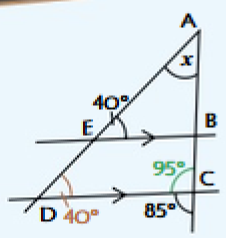
First use rule 3 from p.88:
Angles in a quadrilateral add up to 360°, so the fourth angle in the quadrilateral is
$360° - 78° - 112° - 83° = 87°$
Then use rule 2:
Angles on a straight line add up to 180°.
So $\angle CDE = 180° - 87° = 93°$

Parallel Lines and Angle Rules

Sometimes you'll come across questions combining parallel lines and the five angle rules. These look pretty tricky, but like always, just work out all the angles you can find until you get the one you want.

EXAMPLE: Find the value of angle x on the diagram below.

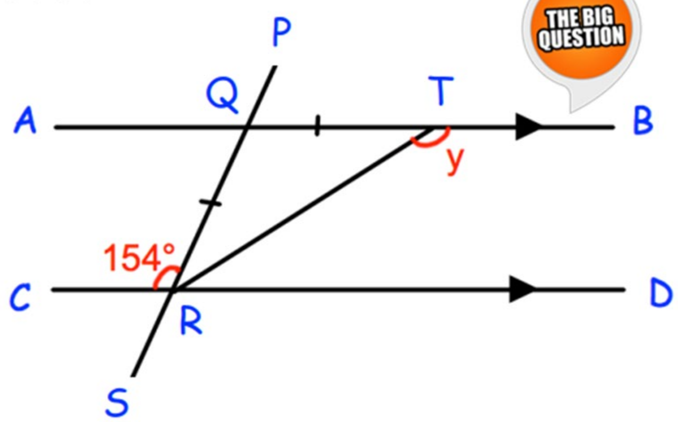
It's always a good idea to label your diagram as you work out each angle.



$\angle AEB$ and $\angle ADC$ are corresponding angles, so they are equal. $\angle ADC = 40°$
Use rule 2 from p.88 to find $\angle ACD$:
Angles on a straight line add up to 180°.
So $\angle ACD = 180° - 85° = 95°$
Use rule 1 to find x:
Angles in a triangle add up to 180°.
So $x = 180° - 95° - 40° = 45°$

THE BIG QUESTION

AB is parallel to CD.



Work out the size of angle y.
Give reasons for your answer.

RQT = 154° alternate angles are equal
QTR = 13° base angles in an isosceles triangle are equal
y = 167° angles on a straight line sum to 180°

Angles in Polygons

A polygon is a many-sided shape, and can be regular or irregular. A regular polygon (p.72) is one where all the sides and angles are the same. By the end of this page you'll be able to work out the angles in them. Wowzers.

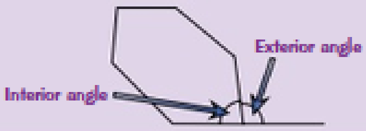
Exterior and Interior Angles

You need to know what exterior and interior angles are and how to find them.

For ANY POLYGON (regular or irregular):

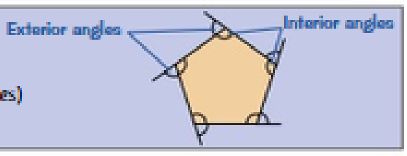
SUM OF EXTERIOR ANGLES = 360°

INTERIOR ANGLE = 180° – EXTERIOR ANGLE



For REGULAR POLYGONS only:

EXTERIOR ANGLE $= \frac{360°}{n}$ (n is the number of sides)



EXAMPLE: Find the exterior and interior angles of a regular octagon.

Octagons have 8 sides: exterior angle $= \frac{360°}{n} = \frac{360°}{8} = 45°$

Use the exterior angle to find the interior angle: interior angle = 180° – exterior angle
= 180° – 45° = 135°

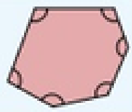
The Tricky One — Sum of Interior Angles

This formula for the sum of the interior angles works for ALL polygons, even irregular ones:

SUM OF INTERIOR ANGLES = (n – 2) × 180°

EXAMPLE: Find the sum of the interior angles of the polygon on the right.

The polygon is a hexagon, so n = 6: Sum of interior angles = (n – 2) × 180°
= (6 – 2) × 180° = 720°



Don't panic if those pesky examiners put algebra in an interior angle question. It looks worse than it is.

EXAMPLE: Find the value of x in the diagram on the right.

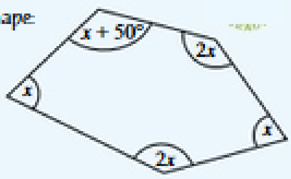
First, find the sum of the interior angles of the 5-sided shape:

Sum of interior angles = (n – 2) × 180°
= (5 – 2) × 180° = 540°

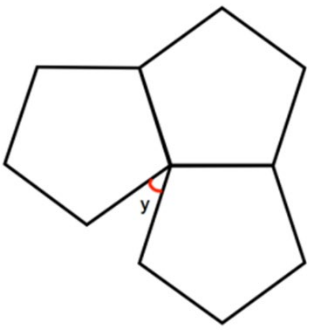
Now write an equation and solve it to find x:

$2x + x + 2x + x + (x + 50°) = 540°$

$7x + 50° = 540° \rightarrow 7x = 490° \rightarrow x = 70°$



THE BIG QUESTION



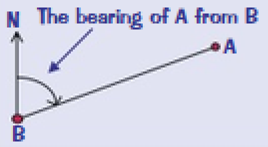
Three identical regular pentagons are joined as shown above.
Work out the size of angle y.

Angles in a pentagon sum to 540°.
Interior angle in a pentagon = 540÷5 = 108°.
y = 360 – 2x108 = 144°

Bearings

Bearings. They'll be useful next time you're off sailing. Or in your Maths exam.

Bearings



1) A bearing is just a direction given as an angle in degrees.
2) All bearings are measured clockwise from the North line.
3) All bearings are given as 3 figures:
   e.g. 060° rather than just 60°, 020° rather than 20° etc.

The 3 Key Words

To find or draw a bearing you must remember three key words:

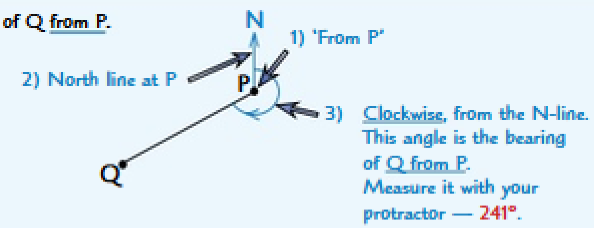
1 'FROM' — Find the word 'FROM' in the question, and put your pencil on the diagram at the point you are going 'from'.

2 NORTH LINE — At the point you are going FROM, draw in a NORTH LINE.

3 CLOCKWISE — Now draw in the angle CLOCKWISE from the NORTH LINE to the line joining the two points — this angle is the bearing.

EXAMPLES:

1. Find the bearing of Q from P.

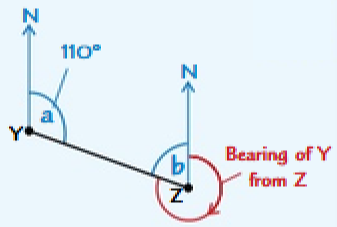


2. The bearing of Z from Y is 110°.
   Find the bearing of Y from Z.

See page 89 for allied angles.

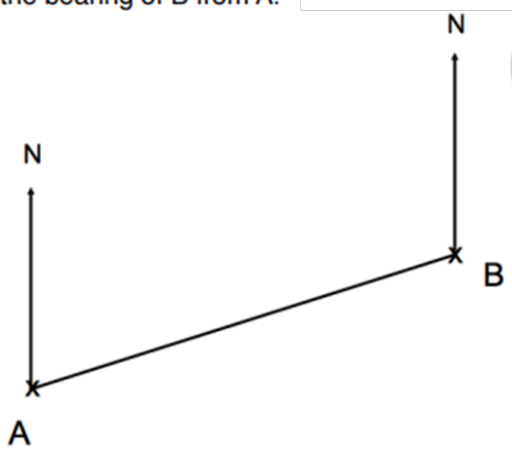
First sketch a diagram so you can see what's going on.
Angles a and b are allied, so they add up to 180°.

Angle b = 180° – 110° = 70°
So bearing of Y from Z = 360° – 70° = 290°.



THE BIG QUESTION

The diagram shows the position of two houses, A and B, on a map.
Measure the bearing of B from A.



075°

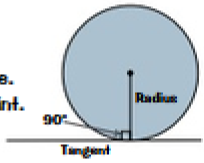
Circle Geometry

After lulling you into a false sense of security with a nice easy page on shapes, it's time to plunge you into the depths of mathematical peril with a 2-page extravaganza on circle theorems. Sorry.

9 ~~Simple~~ Rules to Learn H

1) A TANGENT and a RADIUS meet at 90°.

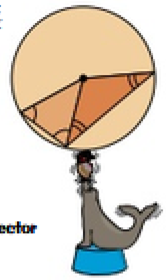
A TANGENT is a line that just touches a single point on the circumference of a circle. A tangent always makes an angle of exactly 90° with the radius it meets at this point.



2) TWO RADII form an ISOSCELES TRIANGLE.

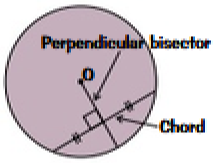
Radii is the plural of radius.

Unlike other isosceles triangles they don't have the little tick marks on the sides to remind you that they are the same — the fact that they are both radii is enough to make it an isosceles triangle.



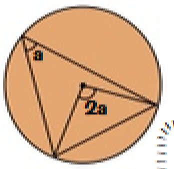
3) The PERPENDICULAR BISECTOR of a CHORD passes through the CENTRE of the circle.

A CHORD is any line drawn across a circle. And no matter where you draw a chord, the line that cuts it exactly in half (at 90°), will go through the centre of the circle.



4) The angle at the CENTRE of a circle is TWICE the angle at the CIRCUMFERENCE.

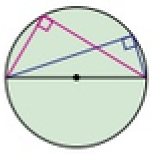
The angle subtended at the centre of a circle is EXACTLY DOUBLE the angle subtended at the circumference of the circle from the same two points (two ends of the same chord).



'Angle subtended at' is just a posh way of saying 'angle made at'.

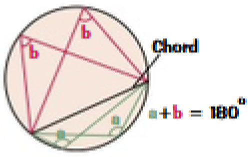
5) The ANGLE in a SEMICIRCLE is 90°.

A triangle drawn from the two ends of a diameter will ALWAYS make an angle of 90° where it hits the circumference of the circle, no matter where it hits.



6) Angles in the SAME SEGMENT are EQUAL.

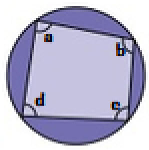
All triangles drawn from a chord will have the same angle where they touch the circumference. Also, the two angles on opposite sides of the chord add up to 180°.

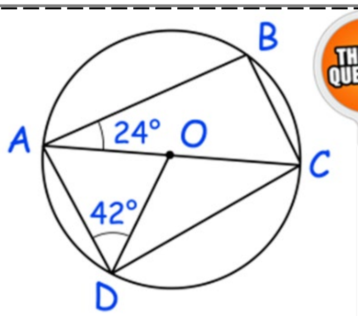


7) OPPOSITE ANGLES in a CYCLIC QUADRILATERAL add up to 180°.

A cyclic quadrilateral is a 4-sided shape with every corner touching the circle. Both pairs of opposite angles add up to 180°.

$a + c = 180°$
$b + d = 180°$





THE BIG QUESTION

(a) Find the size of angle CAD.

(b) Find the size of angle ACB.

(c) Find the size of angle BCD.

a) 042°
b) 66°
c) 114°

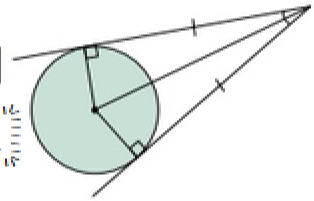
More circle theorems? But I've had enough. Can't I go home now?

Final 2 Rules to Learn

8) TANGENTS from the SAME POINT are the SAME LENGTH.

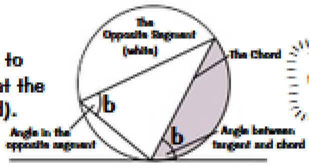
Two tangents drawn from an outside point are always equal in length, creating two congruent right-angled triangles as shown.

There's more about congruence on p.78.



9) The ALTERNATE SEGMENT THEOREM.

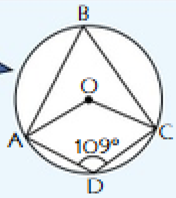
The angle between a tangent and a chord is always equal to 'the angle in the opposite segment' (i.e. the angle made at the circumference by two lines drawn from ends of the chord).



This is probably the hardest rule, so take care.

Using the Circle Theorems

EXAMPLE: A, B, C and D are points on the circumference of the circle, and O is the centre of the circle. Angle ADC = 109°. Work out the size of angles ABC and AOC.



You'll probably have to use more than one rule to solve circle theorem questions — here, ABCD is a cyclic quadrilateral so use rule 7:

7) OPPOSITE ANGLES in a CYCLIC QUADRILATERAL add up to 180°.

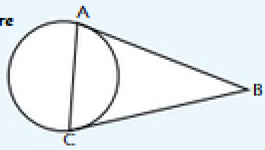
Angles ADC and ABC are opposite, so angle ABC = 180° – 109° = 71°.

Now, angles ABC (which you've just found) and AOC both come from chord AC, so you can use rule 4:

4) The angle at the CENTRE of a circle is TWICE the angle at the CIRCUMFERENCE.

So angle AOC is double angle ABC, which means angle AOC = 71° × 2 = 142°.

EXAMPLE: The diagram shows the triangle ABC, where lines BA and BC are tangents to the circle. Show that line AC is NOT a diameter.



If AC was a diameter passing through the centre, O, then OA and OC would be radii, and angle CAB = angle ACB = 90° by rule 1:

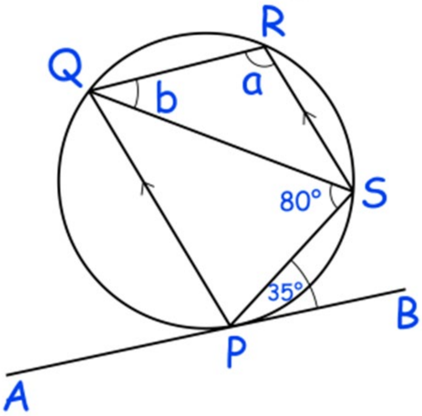
1) A TANGENT and a RADIUS meet at 90°.

However, this would mean that ABC isn't a triangle as you can't have a triangle with two 90° angles, so AC cannot be a diameter.

If angles CAB and ACB were 90°, lines AB and BC would be parallel so would never meet.

PQRS is a cyclic quadrilateral.
APB is a tangent to the circle at P.
PQ is parallel to SR.
Angle SPB = 35° and angle PSQ = 80°

THE BIG QUESTION



Work out the size of angle QRS.
Work out the size of angle RQS.

RQS = 30°
QRS = 115°

Deduction

What do I need to be able to do?

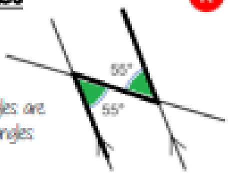
By the end of this unit you should be able to:

- Identify angles in parallel lines
- Solve angle problems
- Make conjectures with angles
- Make conjectures with shapes

Keywords

Parallel: two straight lines that never meet with the same gradient
Perpendicular: two straight lines that meet at 90°
Transversal: a line that crosses at least two other lines
Sum: the result of adding two or more numbers
Conjecture: a statement that might be true but is not proven
Equation: a statement that says two things are equal
Polygon: a 2D shape made from straight edges
Counterexample: an example that disproves a statement

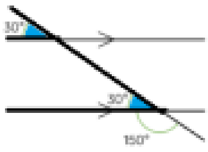
Alternate angles (R)



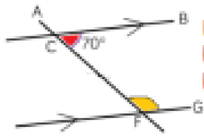
Because alternate angles are equal the highlighted angles are the same size

Corresponding angles (R)

Because corresponding angles are equal the highlighted angles are the same size



Co-interior angles (R)



Because co-interior angles have a sum of 180° the highlighted angle is 110°

As angles on a line add up to 180° co-interior angles can also be calculated from applying alternate/ corresponding rules first

Solving angle problems

Angles on a straight Line
180°

Link angle facts to alg

Form an equation

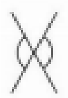
$2x + 4x = 180°$

State the reason

The sum of angles on a straight line is 180°

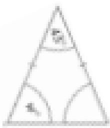
Solve

$2x + 4x = 180°$
$6x = 180°$
$x = 30°$



Vertically opposite angles
Equal

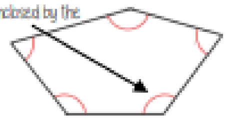
Angles around a point
360°



Triangles
Sum of angles is 180°

Isosceles have the same base angles

Interior Angles
The angles enclosed by the polygon



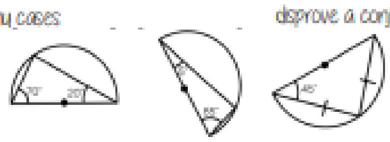
(number of sides – 2) x 180

Making conjectures with angles

True | Always | Never | False | Sometimes

Proving a conjecture
A pattern is noticed for many cases

Disproving a conjecture
Only one counterexample is needed to disprove a conjecture



Apply the angle rules — The sum of angles in a triangle is 180°

Test the theory
$180 - 70 - 20 = 90$
$180 - 85 - 5 = 90$
$180 - 45 - 45 = 90$

Make conjecture — The angle that meets the circumference in a semi circle is 90°

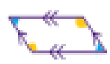
Making conjectures with shapes

Keywords and facts to recall with shape

Area: the amount of space inside a shape
Perimeter: the length around a shape
Regular Polygons: All sides and angles are equal

Quadrilateral Facts

Square
All sides equal size
All angles 90°
Opposite sides are parallel



Parallelogram
Opposite sides are parallel
Opposite angles are equal
Co-interior angles

Rectangle
All angles 90°
Opposite sides are parallel

Kite
No parallel lines
Equal lengths on top sides
Equal lengths on bottom sides
One pair of equal angles

Rhombus
All sides equal size
Opposite angles are equal

Proportion

What do I need to be able to do?

By the end of this unit you should be able to:

- Compare quantities using ratio
- Link ratios and fractions and make comparisons
- Share in a given ratio
- Link Ratio and scales and graphs
- Solve problems with currency conversions
- Solve 'best buy' problems
- Combine ratios

Keywords

Ratio: a statement of how two numbers compare
Equivalent: of equal value
Proportion: a statement that links two ratios
Integer: whole number, can be positive, negative or zero
Fraction: represents how many parts of a whole
Denominator: the number below the line on a fraction. The number represent the total number of parts
Numerator: the number above the line on a fraction. The top number. Represents how many parts are taken
Origin: (0,0) on a graph. The point the two axes cross
Gradient: The steepness of a line

Compare with ratio (R)

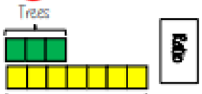
'For every dog there are 2 cats'

The ratio has to be written in the same order as the information is given
e.g. 2:1 I would represent 2 dogs for every 1 cat

Units have the be of the same value to compare ratios

Ratios and fraction (R)

Trees : Flowers
3 : 7



Fraction of trees

$\frac{\text{Number of parts of in group}}{\text{Total number of parts}} = \frac{3}{10}$

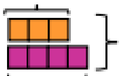
Fraction

Sharing a whole into a given ratio (R)

James and Lucy share £350 in the ratio 3:4
Work out how much each person earns

Model the Question

James : Lucy
3 : 4



Find the value of one part

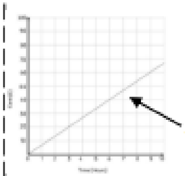
Whole: £350
7 parts to share between (3 James, 4 Lucy)
£350 ÷ 7 = £50
□ = one part = £50

Put back into the question

James = 3 x £50 = £150

Lucy = 4 x £50 = £200

Ratio and graphs (R)



Graphs with a constant ratio are directly proportional

- Form a straight line
- Pass through (0,0)

The gradient is the constant ratio

Ratio and scale (R)

A picture of a car is drawn with a scale of 1:30

The car image is 10cm

Image : Real life
÷10 1cm : 30cm x10
x10 10cm : 300cm

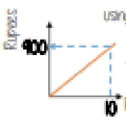
Conversion between currencies (R)

For every £1 I have 90 Rupees

£1 = 90 Rupees ← Currency is directly proportional

÷10 £1 = 90 Rupees ÷10
x10 £10 = 900 Rupees x10

Currency can be converted using a conversion graph



Convert 630 Rupees into Pounds

x7 £1 = 90 Rupees x7 ← 630 ÷ 90 = 7
£7 = **630 Rupees**

Ratios in 1:n and n:1

This is asking you to cancel down until the part indicated represents 1

Show the ratio 4:20 in the ratio of 1:n

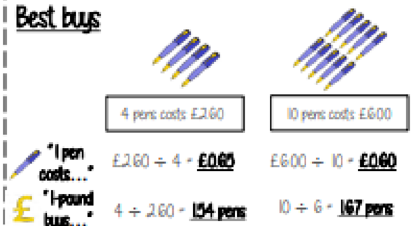
The question states that **this part has to be 1 unit**. Therefore **Divide by 4**

4 : 20
1 : 5

This side has to be divided by 4 too – to keep in **proportion**

the n part does not have to be an integer for this type of question

Best buys



	4 pens costs £2.60	10 pens costs £6.00
"1 pen costs…"	£2.60 ÷ 4 = **£0.65**	£6.00 ÷ 10 = **£0.60**
"1 pound buys…"	4 ÷ 2.60 = **1.54 pens**	10 ÷ 6 = **1.67 pens**

You could work out how much 40 pens are and then compare

Compare the solution in the context of the question

The best value has the lowest cost "per pen"

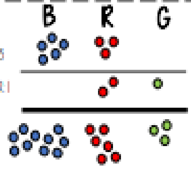
The best value means £1 buys you more pens

Combining ratios

The ratio of Blue counters to Red counters is 5:3

The ratio of Red counters to Green counters is 2:1

Ratio of Blue to Red to Green



Use equivalent ratios to allow comparison of the group that is common to both statements

Lowest common multiple of the ratio both statements share

In a box there are white chocolates, milk chocolates and dark chocolates.

The ratio of white chocolates to milk chocolates is 3:5
The ratio of milk chocolates to dark chocolates is 8:1

What fraction of the chocolates are white chocolate?

23
8

If you were worried I was running out of great stuff to say about ratios then worry no more...

Changing Ratios

You'll need to know how to deal with all sorts of questions where the ratio changes.
Have a look at the examples to see how to handle them.

EXAMPLE: In an animal sanctuary there are 20 peacocks, and the ratio of peacocks to pheasants is 4:9. If 5 of the pheasants fly away, what is the new ratio of peacocks to pheasants? Give your answer in its simplest form.

1) Find the original number of pheasants.

peacocks : pheasants

$4:9$ → ×5 → $20:45$

2) Work out the number of pheasants remaining.

$45 - 5 = 40$ pheasants left

3) Write the new ratio of peacocks to pheasants and simplify.

peacocks : pheasants

$20:40$ → ÷20 → $1:2$

EXAMPLE: (H) The ratio of male to female pupils going on a skiing trip is 5:3. Four male teachers and nine female teachers are also going on the trip. The ratio of males to females going on the trip is 4:3 (including teachers). How many female pupils are going on the trip?

1) WRITE THE RATIOS AS EQUATIONS

Let m be the number of male pupils and f be the number of female pupils.

$m:f = 5:3$

$(m + 4):(f + 9) = 4:3$

2) TURN THE RATIOS INTO FRACTIONS (see p.59)

$\frac{m}{f} = \frac{5}{3}$ and $\frac{m+4}{f+9} = \frac{4}{3}$

3) SOLVE THE TWO EQUATIONS SIMULTANEOUSLY.

$3m = 5f$ and $3m + 12 = 4f + 36$

$3m - 4f = 24$

$-\ \ 3m - 5f = 0$

$f = 24$

24 female pupils are going on the trip.

See pages 37-38 for more on simultaneous equations.

The ratio of the red cards to black cards in a deck is 3:10
2 more red cards are added to the deck.
The ratio of red cards to black cards is now 1:3

Work out the number of black cards in the deck.

THE BIG QUESTION

On 1st March 2001, the ratio of Freddie's age to his mother's age was 1:11
On 1st March 2018, the ratio of Freddie's age to his mother's age was 2:5
Write the ratio of Freddie's age to his mother's age on 1st March 2030

If you were worried I was running out of great stuff to say about ratios then worry no more…

Proportional Division

In a proportional division question a TOTAL AMOUNT is split into parts in a certain ratio.
The key word here is PARTS — concentrate on 'parts' and it all becomes quite painless:

EXAMPLE: Jess, Mo and Greg share £9100 in the ratio 2:4:7. How much does Mo get?

1) ADD UP THE PARTS:
The ratio 2:4:7 means there will be a total of 13 parts: $2 + 4 + 7 = 13$ parts

2) DIVIDE TO FIND ONE "PART":
Just divide the total amount by the number of parts: £9100 ÷ 13 = £700 (= 1 part)

3) MULTIPLY TO FIND THE AMOUNTS:
We want to know Mo's share, which is 4 parts: 4 parts = 4 × £700 = £2800

Watch out for pesky proportional division questions that don't give you the total amount.
You can't just follow the method above, you'll have to be a bit more crafty.

EXAMPLE: A baguette is cut into 3 pieces. The second piece is twice as long as the first and the third piece is five times as long as the first.

a) Find the ratio of the lengths of the 3 pieces. Give your answer in its simplest form.
If the first piece is 1 part, then the second piece is 1 × 2 = 2 parts
and the third piece is 1 × 5 = 5 parts. So the ratio of the lengths = 1:2:5.

b) The first piece is 28 cm smaller than the third piece. How long is the second piece?
1) Work out how many parts 28 cm makes up. 28 cm = 3rd piece – 1st piece = 5 parts – 1 part = 4 parts
2) Divide to find one part. 28 cm ÷ 4 = 7 cm
3) Multiply to find the length of the 2nd piece. 2nd piece = 2 parts = 2 × 7 cm = 14 cm

THE BIG QUESTION

The angles in a triangle are in the ratio 1:1:4

(a) Find the size of each angle
(b) What type of triangle is it?

Flour, sugar and butter are mixed in the ratio 6:2:3
How many grams of flour and sugar are needed to mix with 180g of butter?

The ratio of Mollie's age to Heather's age is 4:9
Heather is 40 years older than Mollie
How old is Mollie?

30° 30° 120° isosceles
60g sugar 90g butter
32

Direct and Inverse Proportion

Direct proportion problems all involve amounts that increase or decrease together. Awww.

Learn the Golden Rule for Proportion Questions

There are lots of exam questions which at first sight seem completely different but in fact they can all be done using the GOLDEN RULE...

DIVIDE FOR ONE, THEN TIMES FOR ALL

EXAMPLE: 5 pints of milk cost £1.30. How much will 3 pints cost?

The GOLDEN RULE tells you to:
Divide the price by 5 to find how much FOR ONE PINT, then multiply by 3 to find how much FOR 3 PINTS.

1 pint: £1.30 ÷ 5 = 0.26 = 26p
3 pints: 26p × 3 = 78p

EXAMPLE: Emma is handing out some leaflets. She gets paid per leaflet she hands out. If she hands out 300 leaflets she gets £2.40. How many leaflets will she have to hand out to earn £8.50?

Divide by £2.40 to find how many leaflets she has to hand out to earn £1.
To earn £1: 300 ÷ £2.40 = 125 leaflets

Multiply by £8.50 to find how many leaflets she has to hand out to earn £8.50.
To earn £8.50: 125 × £8.50 = 1062.5
So she'll need to hand out 1063 leaflets.

You need to round your answer up because 1062 wouldn't be enough.

Scaling Recipes Up or Down

EXAMPLE: Judy is making orange and pineapple punch using the recipe shown on the right. She wants to make enough to serve 20 people. How much of each ingredient will Judy need?

Fruit Punch (serves 8)
800 ml orange juice
140 g fresh pineapple

The GOLDEN RULE tells you to divide each amount by 8 to find how much FOR ONE PERSON, then multiply by 20 to find how much FOR 20 PEOPLE.

So for 1 person you need:
800 ml ÷ 8 = 100 ml orange juice ⇒ 20 × 100 ml = 2000 ml orange juice
140 g ÷ 8 = 17.5 g pineapple ⇒ 20 × 17.5 g = 350 g pineapple

And for 20 people you need:
2000 ml orange juice
350 g pineapple

THE BIG QUESTION

Tia uses this recipes to make hot cross buns.
Tia is going to use this recipe to make 9 hot cross buns.

makes 12
480g flour
60g caster sugar
200ml milk
1 egg
50g butter
100g currant

(a) How much of each ingredient does Tia need?

Grace uses the same recipe.
She uses 500ml of milk.

(b) How many hot cross buns is Grace making?

a) 360g flour, 45g caster sugar, 150ml milk, 0.75 egg, 37.5g butter
b) 30

There can sometimes be a lot of information packed into proportion questions, but the method of solving them always stays the same — have a look at this page and see what you think.

Direct Proportion

1) Two quantities, A and B, are in direct proportion (or just in proportion) if increasing one increases the other one proportionally. So if quantity A is doubled (or trebled, halved, etc.), so is quantity B.
2) Remember this golden rule for direct proportion questions:

DIVIDE for ONE, then TIMES for ALL

EXAMPLE: Hannah pays £3.60 per 400 g of cheese. She uses 220 g of cheese to make 4 cheese pasties. How much would the cheese cost if she wanted to make 50 cheese pasties?

In 1 pasty there is: 220 g ÷ 4 = 55 g of cheese
So in 50 pasties there is: 55 g × 50 = 2750 g of cheese
1 g of cheese would cost: £3.60 ÷ 400 = 0.9p
So 2750 g of cheese would cost: 0.9 × 2750 = 2475p = £24.75

There will often be lots of stages to direct proportion questions — keep track of what you've worked out at each stage.

Inverse Proportion

1) Two quantities, C and D, are in inverse proportion if increasing one quantity causes the other quantity to decrease proportionally. So if quantity C is doubled (or tripled, halved, etc.), quantity D is halved (or divided by 3, doubled etc.).
2) The rule for finding inverse proportions is:

TIMES for ONE, then DIVIDE for ALL

EXAMPLE: 4 bakers can decorate 100 cakes in 5 hours.

a) How long would it take 10 bakers to decorate the same number of cakes?

100 cakes will take 1 baker: 5 × 4 = 20 hours
So 100 cakes will take 10 bakers: 20 ÷ 10 = 2 hours for 10 bakers

b) How long would it take 11 bakers to decorate 220 cakes?

100 cakes will take 1 baker: 20 hours
1 cake will take 1 baker: 20 ÷ 100 = 0.2 hours
220 cakes will take 1 baker: 0.2 × 220 = 44 hours
220 cakes will take 11 bakers: 44 ÷ 11 = 4 hours

The number of bakers is inversely proportional to number of hours — but the number of cakes is directly proportional to the number of hours.

It takes 5 machines 6 hours to produce 1000 DVDs. Work out how long it would take 4 machines to produce 2000 DVDs.

THE BIG QUESTION

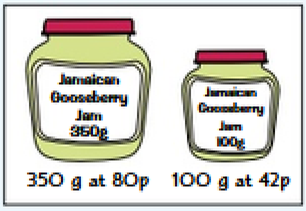
15 hours

Best Buy Questions

A slightly different type of direct proportion question is comparing the 'value for money' of 2 or 3 similar items. For these, follow the second GOLDEN RULE...

Divide by the PRICE in pence (to get the amount per penny)

EXAMPLE: The local 'Supplies 'n' Vittals' stocks two sizes of Jamaican Gooseberry Jam, as shown on the right. Which of these represents better value for money?



Follow the GOLDEN RULE — divide by the price in pence to get the amount per penny.

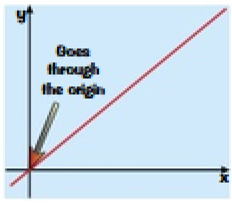
In the 350 g jar you get 350 g ÷ 80p = 4.38 g per penny
In the 100 g jar you get 100 g ÷ 42p = 2.38 g per penny

The 350 g jar is better value for money, because you get more jam per penny.

In some cases it might be easier to divide by the weight to get the cost per gram.
If you're feeling confident then you can do it this way — if not, the golden rule always works.

Graphing Direct Proportion

Two things are in direct proportion if, when you plot them on a graph, you get a straight line through the origin.



Remember, the general equation for a straight line through the origin is $y = Ax$ (see p.43) where A is a number.
All direct proportions can be written as an equation in this form.

EXAMPLE: The amount of petrol, p litres, a car uses is directly proportional to the distance, d km, that the car travels. The car used 12 litres of petrol on a 160 km journey.

a) Write an equation in the form $p = Ad$ to represent this direct proportion.

1) Put the values of $p = 12$ and $d = 160$ into the equation to find the value of A.

$12 = A \times 160$

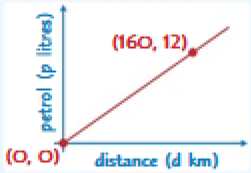
$A = \frac{12}{160}$

$A = 0.075$

2) Put the value of A back into the equation.

$p = 0.075d$

b) Sketch the graph of this direct proportion, marking two points on the line.



THE BIG QUESTION

A cereal bar is sold in packs of 4, 6 or 8.

The 4 pack of cereal bars costs £1.80 and it is the least value for money.
The 8 pack of cereal bars cost £3.52 and it is the best value for money.

Work out (a) the lowest price of the 6 pack of cereal bar
(b) the highest price of the 6 pack of cereal bar

a) £2.65
b) £2.69

Algebraic proportion questions normally involve two variables (often x and y) which are linked in some way.

Types of Proportion (H)

$\propto$ means 'is proportional to'.

1) The simple proportions are 'y is proportional to x' ($y \propto x$) and 'y is inversely proportional to x' ($y \propto \frac{1}{x}$).
2) You can always turn a proportion statement into an equation by replacing '$\propto$' with '= k' like this:

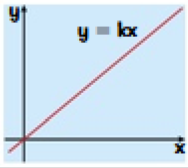
	Proportionality	Equation
'y is proportional to x'	$y \propto x$	$y = kx$
'y is inversely proportional to x'	$y \propto \frac{1}{x}$	$y = \frac{k}{x}$

k is just some constant (unknown number)

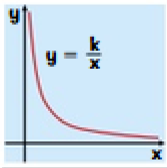
3) Trickier proportions involve y varying proportionally or inversely to some function of x, e.g. x^2, x^3, $\sqrt{x}$ etc.

	Proportionality	Equation
'y is proportional to the square of x'	$y \propto x^2$	$y = kx^2$
't is proportional to the square root of h'	$t \propto \sqrt{h}$	$t = k\sqrt{h}$
'V is inversely proportional to r cubed'	$V \propto \frac{1}{r^3}$	$V = \frac{k}{r^3}$

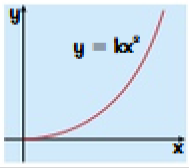
4) Once you've written the proportion statement as an equation you can easily graph it.



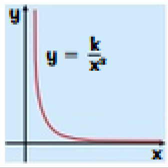
y is proportional to x



y is inversely proportional to x



y is proportional to x^2



y is inversely proportional to x^3

Handling Algebra Questions on Proportion (H)

1) Write the sentence as a proportionality and replace '$\propto$' with '= k' to make an equation (as above).
2) Find a pair of values (x and y) somewhere in the question — substitute them into the equation to find k.
3) Put the value of k into the equation and it's now ready to use, e.g. $y = 3x^2$.
4) Inevitably, they'll ask you to find y, having given you a value for x (or vice versa).

EXAMPLE: G is inversely proportional to the square root of H. When G = 2, H = 16. Find an equation for G in terms of H, and use it to work out the value of G when H = 36.

1) Convert to a proportionality and replace $\propto$ with '= k' to form an equation. $G \propto \frac{1}{\sqrt{H}}$ $\quad G = \frac{k}{\sqrt{H}}$
2) Use the values of G and H (2 and 16) to find k. $2 = \frac{k}{\sqrt{16}} = \frac{k}{4} \Rightarrow k = 8$
3) Put the value of k back into the equation. $G = \frac{8}{\sqrt{H}}$ ← This is the equation for G in terms of H.
4) Use your equation to find the value of G. When H = 36, $G = \frac{8}{\sqrt{36}} = \frac{8}{6} = \frac{4}{3}$

An object when dropped, falls *d* metres in *t* seconds.
d is directly proportional to the square of *t*.
The object falls 80 metres in 4 seconds.
Work out how far the object falls in 9 seconds.

THE BIG QUESTION

The number of days, D, to complete research is inversely proportional to the number of researchers, R, who are working.
The research takes 125 days to complete when 24 people work on it.
Find out how many people are needed to complete the research in 60 days.

$d = 5t^2$ $\quad d = 405$

$D = \frac{3000}{R}$ $\quad D = 50$

Converting Units

A nice easy page for a change — just some facts to learn. Hooray!

Metric and Imperial Units

COMMON METRIC CONVERSIONS

1 cm = 10 mm
1 m = 100 cm
1 km = 1000 m
1 kg = 1000 g
1 tonne = 1000 kg
1 litre = 1000 ml
1 litre = 1000 cm^3
1 cm^3 = 1 ml

COMMON IMPERIAL CONVERSIONS

1 Yard = 3 Feet
1 Foot = 12 Inches
1 Gallon = 8 Pints
1 Stone = 14 Pounds
1 Pound = 16 Ounces

You only need to remember the metric conversions, but you should be able to use them all.

COMMON METRIC-IMPERIAL CONVERSIONS

1 kg ≈ 2.2 pounds
1 foot ≈ 30 cm
1 gallon ≈ 4.5 litres
1 mile ≈ 1.6 km

Converting Units

To convert between units, multiply or divide by the conversion factor.

Always check your answer looks sensible — if it's not then chances are you divided instead of multiplying or vice versa.

Converting speeds is a bit trickier because speeds are made up of two measures — a distance and a time. You have to convert the distance unit and the time unit separately.

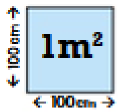
EXAMPLES:

1. Convert 10 pounds into kg.
   2.2 pounds ≈ 1 kg
   So 10 pounds ≈ 10 ÷ 2.2
   ≈ 4.5 kg

2. A rabbit's top speed is 56 km/h. How fast is this in m/s?
   1) First convert from km/h to m/h:
   56 km/h = (56 × 1000) m/h = 56 000 m/h
   2) Now convert from m/h to m/s:
   56 000 m/h = (56 000 ÷ 60 ÷ 60) m/s = 15.6 m/s (1 d.p.)

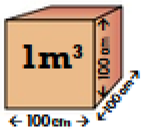
Converting Area and Volume Measurements

Converting areas and volumes from one unit to another is an exam disaster that you have to know how to avoid. 1 m^2 definitely does NOT equal 100 cm^2. Remember this and read on for why.



1 m^2 = 100 cm × 100 cm = 10 000 cm^2
1 cm^2 = 10 mm × 10 mm = 100 mm^2

1 m^3 = 100 cm × 100 cm × 100 cm = 1 000 000 cm^3
1 cm^3 = 10 mm × 10 mm × 10 mm = 1000 mm^3



EXAMPLES:

1. Convert 9 m^2 to cm^2.
   To change area measurements from m^2 to cm^2 multiply by 100 twice.
   9 × 100 × 100 = 90 000 cm^2

2. Convert 60 000 mm^3 to cm^3.
   To change volume measurements from mm^3 to cm^3 divide by 10 three times.
   60 000 ÷ (10 × 10 × 10) = 60 cm^3

Given that

1 gallon = 8 pints and 1 litre = 1.86 pints

Convert 15 gallons to litres.

THE BIG QUESTION

Tommy has been asked to changed $2m^3$ into cm^3
He says:

"since there are 100 centimetres in 1 metres, the answer is $200m^3$"

Explain why Tommy is incorrect.

There are 1,000,000cm^3 in 1 m^3
64.5 litres

Compound Units

Speed, density and pressure. Just a matter of learning the formulas, bunging the numbers in and watching the units.

Speed = Distance ÷ Time

Speed is the distance travelled per unit time, e.g. the number of km per hour or metres per second.

$$\text{SPEED} = \frac{\text{DISTANCE}}{\text{TIME}} \qquad \text{TIME} = \frac{\text{DISTANCE}}{\text{SPEED}} \qquad \text{DISTANCE} = \text{SPEED} \times \text{TIME}$$

Formula triangles are a handy tool for remembering formulas like these. The speed one is shown below.

D
S×T

HOW DO YOU USE FORMULA TRIANGLES?

1) COVER UP the thing you want to find and WRITE DOWN what's left showing.
2) Now PUT IN THE VALUES and CALCULATE — check the UNITS in your answer.

EXAMPLE: A car travels 9 miles at 36 miles per hour. How many minutes does it take?

Write down the formula, put in the values and calculate: $\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{9 \text{ miles}}{36 \text{ mph}} = 0.25 \text{ hours} = 15 \text{ minutes}$

Density = Mass ÷ Volume

Density is the mass per unit volume of a substance. It's usually measured in kg/m^3 or g/cm^3.

$$\text{DENSITY} = \frac{\text{MASS}}{\text{VOLUME}} \qquad \text{VOLUME} = \frac{\text{MASS}}{\text{DENSITY}} \qquad \text{MASS} = \text{DENSITY} \times \text{VOLUME}$$

M
D×V

EXAMPLE: A giant 'Wunda-Choc' bar has a density of 1.3 g/cm^3. If the bar's volume is 1800 cm^3, what is the mass of the bar in kg?

Write down the formula, put in the values and calculate: mass = density × volume = 1.3 g/cm^3 × 1800 cm^3 = 2340 g = 2.34 kg

CHECK YOUR UNITS MATCH If the density is in g/cm^3, the volume must be in cm^3 and you'll get a mass in g.

Pressure = Force ÷ Area

'N' stands for 'Newtons'.

Pressure is the amount of force acting per unit area. It's usually measured in N/m^2, or pascals (Pa).

$$\text{PRESSURE} = \frac{\text{FORCE}}{\text{AREA}} \qquad \text{AREA} = \frac{\text{FORCE}}{\text{PRESSURE}} \qquad \text{FORCE} = \text{PRESSURE} \times \text{AREA}$$

F
P×A

EXAMPLE: A cylindrical barrel with a weight of 200 N rests on horizontal ground. The radius of the circular face resting on the ground is 0.4 m. Calculate the pressure exerted by the barrel on the ground to 1 d.p.

Work out the area of the circular face: $\pi \times 0.4^2 = 0.5026... \text{ m}^2$

Write down the pressure formula, put in the values and calculate: $\text{pressure} = \frac{\text{force}}{\text{area}} = \frac{200 \text{ N}}{0.5026... \text{ m}^2} = 397.8873... \text{ N/m}^2 = 397.9 \text{ N/m}^2$ (1 d.p.)

THE BIG QUESTION

The mass of $4m^3$ of silver is 41960kg.
The density of gold is $19300kg/m^3$.

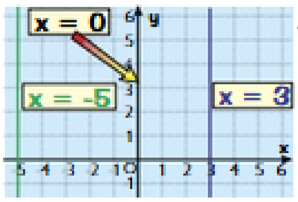
Calculate the difference in mass between $5m^3$ of silver and $5m^3$ of gold.

44,050kg

Straight Line Graphs

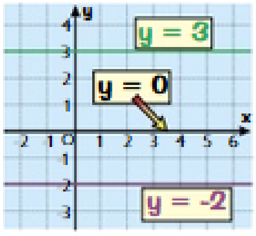
If you thought I-spy was a fun game, wait 'til you play 'recognise the straight-line graph from the equation'.

Vertical and Horizontal lines: '$x = a$' and '$y = a$'

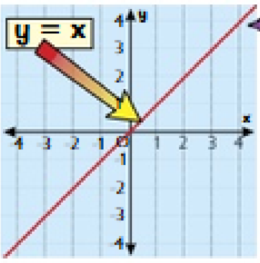


$x = a$ is a vertical line through 'a' on the x-axis

$y = a$ is a horizontal line through 'a' on the y-axis

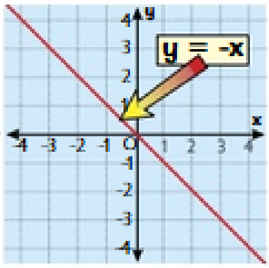


The Main Diagonals: '$y = x$' and '$y = -x$'



'$y = x$' is the main diagonal that goes UPHILL from left to right.

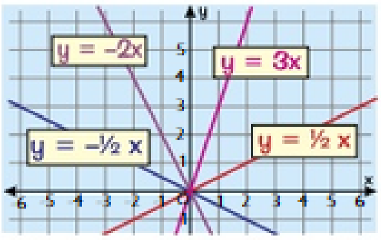
'$y = -x$' is the main diagonal that goes DOWNHILL from left to right.



Other Lines Through the Origin: '$y = ax$' and '$y = -ax$'

$y = ax$ and $y = -ax$ are the equations for A SLOPING LINE THROUGH THE ORIGIN.

The value of 'a' (known as the gradient) tells you the steepness of the line. The bigger 'a' is, the steeper the slope. A MINUS SIGN tells you it slopes DOWNHILL.

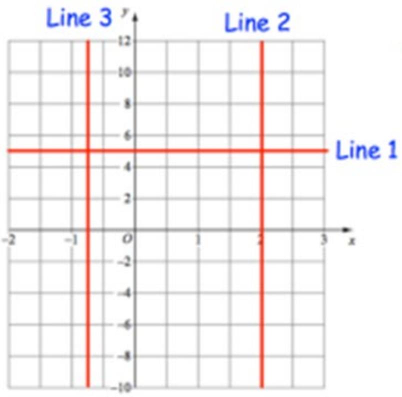


Learn to Spot Straight Lines from their Equations

All straight-line equations just contain 'something x, something y and a number'.

Straight lines:	
$x - y = 0$	$y = 2 + 3x$
$2y - 4x = 7$	$4x - 3 = 5y$

NOT straight lines:	
$y = x^3 + 3$	$\frac{1}{y} + \frac{1}{x} = 2$
$x^2 = 4 - y$	$xy + 3 = 0$



Write down the equation of

(a) Line 1

(b) Line 2

(c) Line 3

THE BIG QUESTION

(a) y =5
(b) x = 2
(c) x = -0.75

You might be asked to DRAW THE GRAPH of an equation in the exam.
This EASY METHOD will net you the marks every time:

1) **Choose 3 values of x and draw up a wee table.**
2) **Work out the corresponding y-values.**
3) **Plot the coordinates, and draw the line.**

You might get lucky and be given a table in an exam question. Don't worry if it contains 5 or 6 values.

Doing the 'Table of Values'

EXAMPLE: Draw the graph of $y = 2x - 3$ for values of x from -2 to 4.

1. **Choose 3 easy x-values for your table:**
   Use x-values from the grid you're given.
   Avoid negative ones if you can.

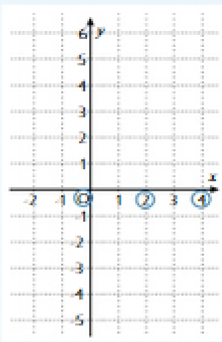
x	0	2	4
y			

2. **Find the y-values** by putting each x-value into the equation:

x	0	2	4
y	−3	1	5

When $x = 0$,
$y = 2x - 3$
$= (2 \times 0) - 3 = -3$

When $x = 4$,
$y = 2x - 3$
$= (2 \times 4) - 3 = 5$



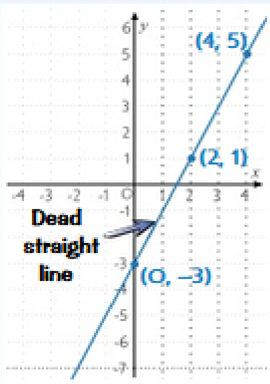
Plotting the Points and Drawing the Graph

EXAMPLE: ...continued from above.

3. **PLOT EACH PAIR of x- and y- values from your table.**
   The table gives the coordinates (0, −3), (2, 1) and (4, 5).

   Now draw a STRAIGHT LINE through your points.

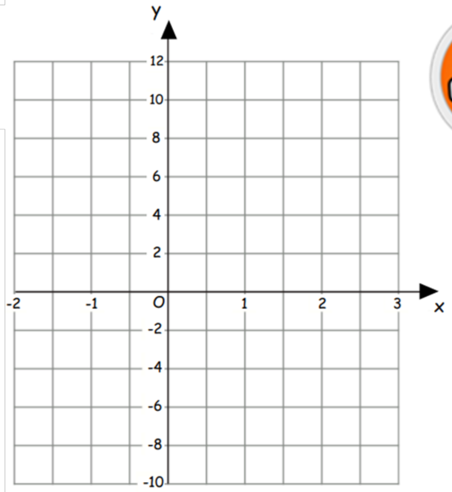
> If one point looks a bit wacky, check 2 things:
> – the y-values you worked out in the table
> – that you've plotted the points properly.



Complete the table of values for y = 2x + 4.

x	-1	0	1	2	3
y		4			10

On the grid, draw the graph of y = 2x + 4 for values of x from –1 to 3



y	2	4	6	8	10
x	-1	0	1	2	3

$y = mx + c$

This sounds a bit scary, but give it a go and you might like it.

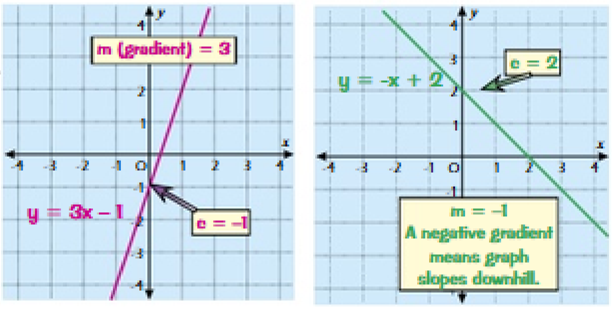
$y = mx + c$ is the Equation of a Straight Line

$y = mx + c$ is the general equation for a straight-line graph, and you need to remember:

'm' is equal to the **GRADIENT** of the graph

'c' is the value **WHERE IT CROSSES THE Y-AXIS** and is called the **Y-INTERCEPT**.

'm' and 'c' are always just numbers — so $y = 3x - 1$ and $y = -x + 2$ are in $y = mx + c$ form.



You might have to rearrange a straight-line equation to get it into this form:

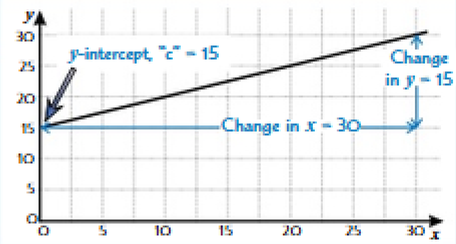
Straight line:		Rearranged into 'y = mx + c'	
$y = 2 + 3x$	→	$y = 3x + 2$	$(m = 3, c = 2)$
$x - y = 4$	→	$y = x - 4$	$(m = 1, c = -4)$
$4 - 3x = y$	→	$y = -3x + 4$	$(m = -3, c = 4)$

WATCH OUT: people mix up 'm' and 'c' when they get something like $y = 5 + 2x$. Remember, 'm' is the number in front of the 'x' and 'c' is the number on its own.

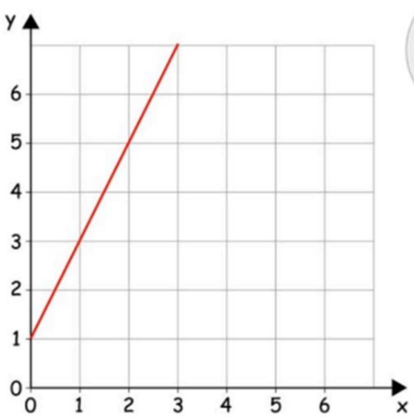
Finding the Equation of a Straight-Line Graph

EXAMPLE: Find the equation of the line on the graph in the form $y = mx + c$.

1. Find 'm' (gradient) $m = \frac{\text{change in } y}{\text{change in } x} = \frac{15}{30} = \frac{1}{2}$
   It's an uphill graph, so the gradient is positive.
2. Read off 'c' (y-intercept) $c = 15$
3. Use these to write the equation in the form $y = mx + c$. $y = \frac{1}{2}x + 15$



A straight line L is shown on the grid.



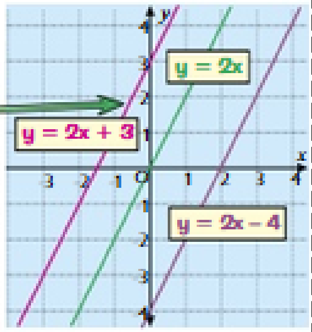
THE BIG QUESTION

Work out the equation of line L

$y = 2x + 1$

This page covers some of the awkward questions you might get asked about straight lines.

Parallel Lines Have the Same Gradient



Parallel lines all have the same gradient, which means their $y = mx + c$ equations all have the same value of m.
So the lines: $y = 2x + 3$, $y = 2x$ and $y = 2x - 4$ are all parallel.

EXAMPLE: Line J has a gradient of –3. Find the equation of Line K, which is parallel to Line J and passes through point (2, 3).

Lines J and K are parallel so their gradients are the same $\Rightarrow m = -3$

$y = -3x + c$

When $x = 2$, $y = 3$:
$3 = (-3 \times 2) + c \Rightarrow 3 = -6 + c$
$c = 9$

$y = -3x + 9$

1) First find the 'm' value for Line K.
2) Substitute the value for 'm' into $y = mx + c$ to give you the 'equation so far'.
3) Substitute the x and y values for the given point on Line K and solve for 'c'.
4) Write out the full equation.

Finding the Equation of a Line Through Two Points

If you're given two points on a line you can find the gradient, then you can use the gradient and one of the points to find the equation of the line. It's a bit tricky, but try to follow the method used in this example.

EXAMPLE: Find the equation of the straight line that passes through (–2, 9) and (3, –1). Give your answer in the form $y = mx + c$.

1) Use the two points to find 'm' (gradient).

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-1 - 9}{3 - (-2)} = \frac{-10}{5} = -2$$

So $y = -2x + c$

2) Substitute one of the points into the equation you've just found.

Substitute (–2, 9) into eqn: $9 = (-2 \times -2) + c$
$9 = 4 + c$

3) Rearrange the equation to find 'c'.

$c = 9 - 4$
$c = 5$

4) Write out the full equation.

$y = -2x + 5$

Sometimes you'll be asked to give your equation in other forms, such as $ax + by + c = 0$. Just rearrange your $y = mx + c$ equation to get it in this form. It's no biggie.

A line has a gradient of 8 and passes through the point (2, 3).
Find the equation of the line.

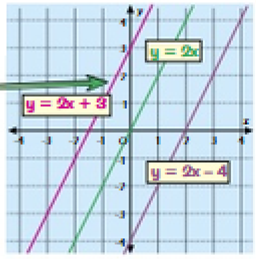
THE BIG QUESTION

y = 8x -13

Parallel and Perpendicular Graphs

On p.44 you saw how to write the equation of a straight line. Well, you also have to be able to write the equation of a line that's parallel or perpendicular to the straight line you're given.

Parallel Lines Have the Same Gradient



Parallel lines all have the same gradient, which means their $y = mx + c$ equations all have the same value of m.
So the lines: $y = 2x + 3$, $y = 2x$ and $y = 2x - 4$ are all parallel.

EXAMPLE: Line J has a gradient of –0.25. Find the equation of Line K, which is parallel to Line J and passes through point (2, 3).

Lines J and K are parallel so their gradients are the same ⇒ $m = -0.25$

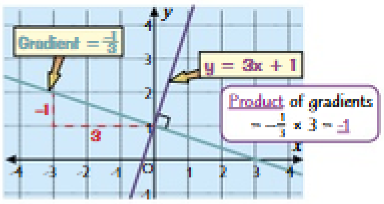
$y = -0.25x + c$

when $x = 2$, $y = 3$:
$3 = (-0.25 \times 2) + c \Rightarrow 3 = -0.5 + c$
$c = 3.5$

$y = -0.25x + 3.5$

1) First find the 'm' value for Line K.
2) Substitute the value for 'm' into $y = mx + c$ to give you the 'equation so far'.
3) Substitute the x and y values for the given point on Line K and solve for 'c'.
4) Write out the full equation.

Perpendicular Line Gradients H



Perpendicular lines cross at a right angle, and if you multiply their gradients together you'll get –1. Pretty nifty that.

If the gradient of the first line is m, the gradient of the other line will be $-\frac{1}{m}$, because $m \times -\frac{1}{m} = -1$.

EXAMPLE: Lines A and B are perpendicular and intersect at (3, 3).
If Line A has the equation $3y - x = 6$, what is the equation of Line B?

Find 'm' (the gradient) for Line A.	$3y - x = 6 \Rightarrow 3y = x + 6$ $\Rightarrow y = \frac{1}{3}x + 2$, so $m_A = \frac{1}{3}$
Find the 'm' value for the perpendicular line (Line B).	$m_B = -\frac{1}{m_A} = -1 \div \frac{1}{3} = -3$
Put this into $y = mx + c$ to give the 'equation so far'.	$y = -3x + c$
Put in the x and y values of the point and solve for 'c'.	$x = 3$, $y = 3$ gives: $3 = (-3 \times 3) + c$ $\Rightarrow 3 = -9 + c \Rightarrow c = 12$
Write out the full equation.	$y = -3x + 12$

THE BIG QUESTION

Write down the equation of each of the following lines

(a) Parallel to $y = 5x - 4$ and passing through (2, 9)

(b) Perpendicular to $y = 2x + 4$ and passing through (0, 3)

a) $y = 5x - 1$
b) $y = -\frac{1}{2}x - 3$

Graphical Inequalities

These questions always involve shading a region on a graph. The method sounds very complicated, but once you've seen it in action with an example, you'll see that it's OK...

Showing Inequalities on a Graph

Here's the method to follow:

1) CONVERT each INEQUALITY to an EQUATION by simply putting an '=' in place of the inequality sign.
2) DRAW THE GRAPH FOR EACH EQUATION — if the inequality sign is < or > draw a dotted line, but if it's ≥ or ≤ draw a solid line.
3) Work out WHICH SIDE of each line you want — put a point (usually the origin) into the inequality to see if it's on the correct side of the line.
4) SHADE THE REGION this gives you.

If using the origin doesn't work (e.g. if the origin lies on the line), just pick another point with easy coordinates and use that instead.

EXAMPLE: Shade the region that satisfies all three of the following inequalities:
$x + y < 5$ $\quad y \le x + 2$ $\quad y > 1.$

1) CONVERT EACH INEQUALITY TO AN EQUATION:
$x + y = 5$, $y = x + 2$ and $y = 1$
2) DRAW THE GRAPH FOR EACH EQUATION (see p.45)
You'll need a dotted line for $x + y = 5$ and $y = 1$ and a solid line for $y = x + 2$.
3) WORK OUT WHICH SIDE OF EACH LINE YOU WANT
This is the fiddly bit. Put $x = 0$ and $y = 0$ (the origin) into each inequality and see if this makes the inequality true or false.

$x + y < 5$:
$x = 0, y = 0$ gives $0 < 5$ which is true.
This means the origin is on the correct side of the line.

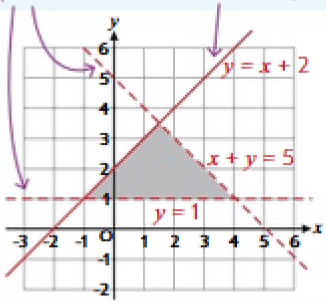
$y \le x + 2$:
$x = 0, y = 0$ gives $0 \le 2$ which is true.
So the origin is on the correct side of this line.

$y > 1$:
$x = 0, y = 0$ gives $0 > 1$ which is false.
So the origin is on the wrong side of this line.

4) SHADE THE REGION
You want the region that satisfies all of these:
— below $x + y = 5$ (because the origin is on this side)
— right of $y = x + 2$ (because the origin is on this side)
— above $y = 1$ (because the origin isn't on this side).

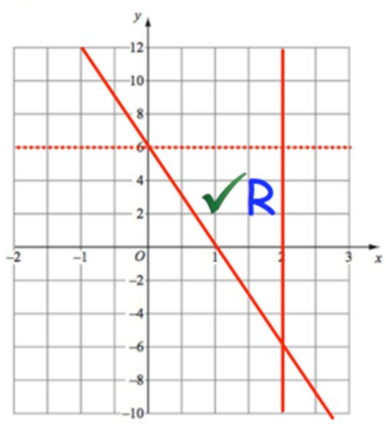
Dotted lines mean the region doesn't include the points on the line.

A solid line means the region does include the points on the line.



Make sure you read the question carefully — you might be asked to label the region instead of shade it, or just mark on points that satisfy all three inequalities. No point throwing away marks because you didn't read the question properly.

State the inequalities that the region labelled R satisfies.

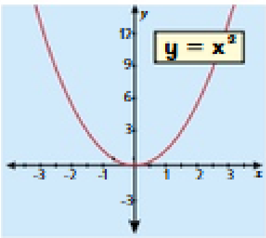


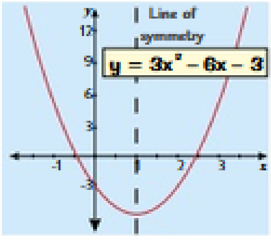
THE BIG QUESTION

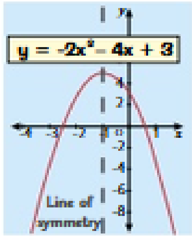
$x < 2$
$y < 6$
$y > -6x + 6$

Quadratic Graphs

Quadratic functions take the form y = anything with x^2 (but no higher powers of x).
x^2 graphs all have the same symmetrical bucket shape.







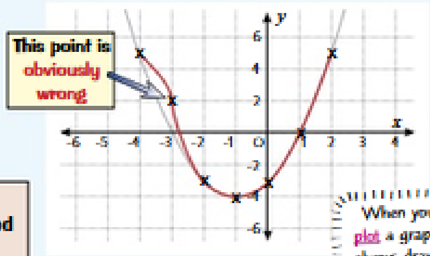
If the x^2 bit has a '−' in front of it then the bucket is upside down.

Plotting Quadratics

EXAMPLE: Complete the table of values for the equation $y = x^2 + 2x - 3$ and then plot the graph.

x	−4	−3	−2	−1	0	1	2
y	5	0	−3	−4	−3	0	5

1) Substitute each x-value into the equation to get each y-value.
   E.g. $y = (-4)^2 + (2 \times -4) - 3 = 5$
2) Plot the points and join them with a completely smooth curve.



NEVER EVER let one point drag your graph off in some ridiculous direction. When a graph is generated from an equation, you never get spikes or lumps.

When you're asked to plot a graph, you should always draw it accurately using this method.

Sketching Quadratics H

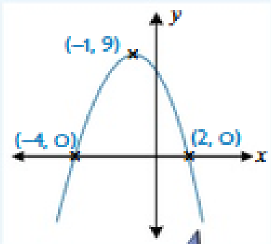
If you're asked to sketch a graph, you won't have to use graph paper or be dead accurate — just find and label the important points and make sure the graph is roughly in the correct position on the axes.

EXAMPLE: Sketch the graph of $y = -x^2 - 2x + 8$, labelling the turning point and x-intercepts with their coordinates.

1 Find all the information you're asked for.

Solve $-x^2 - 2x + 8 = 0$ to find the x-intercepts (see p.34).
$-x^2 - 2x + 8 = -(x + 4)(x - 2) = 0$ so $x = -4$, $x = 2$

Use symmetry to find the turning point of the curve:
The x-coordinate of the turning point is halfway between −4 and 2.
$x = \frac{-4 + 2}{2} = -1$
$y = -(-1)^2 - 2(-1) + 8 = 9$
So the turning point is (−1, 9).



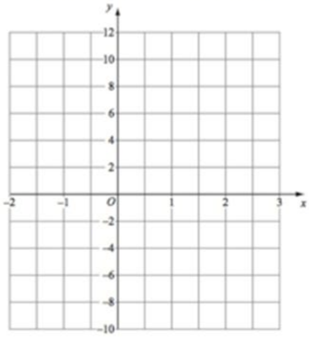
2 Use the information you know to sketch the curve and label the important points.

The x^2 is negative, so the curve is n-shaped.

(a) Complete the table of values for $y = x^2 - 3x + 2$

x	-2	-1	0	1	2	3
y	12				0	

(b) On the grid, draw the graph of $y = x^2 - 3x + 2$ for the values of x from −2 to 3



y	12	6	2	0	0	2
x	−2	−1	0	1	2	3

Quadratic Inequalities

Quadratic inequalities are a bit tricky — you have to remember that there are two solutions (just like quadratic equations), so you might end up with a solution in two separate bits, or an enclosed region.

Take Care with Quadratic Inequalities H

If $x^2 = 4$, then $x = +2$ or -2. So if $x^2 > 4$, $x > 2$ or $x < -2$ and if $x^2 < 4$, $-2 < x < 2$.

As a general rule:

If $x^2 > a^2$ then $x > a$ or $x < -a$
If $x^2 < a^2$ then $-a < x < a$

EXAMPLES:

1. Solve the inequality 25.
   If $x^2 = 25$, then $x = \pm 5$.
   As $x^2 \le 25$, then $-5 \le x \le 5$

2. Solve the inequality $x^2 > 9$.
   If $x^2 = 9$, then $x = \pm 3$.
   As $x^2 > 9$, then $x < -3$ or $x > 3$

If you're confused by the '$x < -3$' bit, try putting some numbers in. E.g. $x = -4$ gives $x^2 = 16$, which is greater than 9, as required.

3. Solve the inequality $3x^2 \ge 48$.
   $(\div 3)$ $\frac{3x^2}{3} \ge \frac{48}{3}$
   $x^2 \ge 16$
   $x \le -4$ or $x \ge 4$

4. Solve the inequality $-2x^2 + 8 > 0$.
   (-8) $-2x^2 + 8 - 8 > 0 - 8$
   $-2x^2 > -8$
   $(\div -2)$ $-2x^2 \div -2 < -8 \div -2$
   $x^2 < 4$
   $-2 < x < 2$

You're dividing by a negative number, so flip the sign.

Sketch the Graph to Help You H

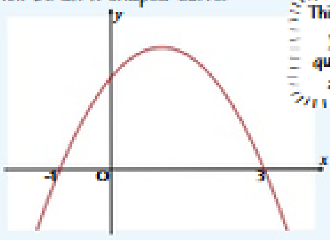
Worst case scenario — you have to solve a ---tic inequality such as $-x^2 + 2x + 3 > 0$. Don't panic — you can use the graph of the quadratic to h re's more on sketching quadratic graphs on p.48).

EXAMPLE: Solve the inequality $-x^2 + 2x + 3 > 0$.

1) Start off by setting the quadratic equal to 0 and factorising:
   $-x^2 + 2x + 3 = 0$
   $x^2 - 2x - 3 = 0$
   $(x - 3)(x + 1) = 0$

2) Now solve the equation to see where it crosses the x-axis:
   $(x - 3)(x + 1) = 0$
   $(x - 3) = 0$, so $x = 3$
   $(x + 1) = 0$, so $x = -1$

3) Then sketch the graph — it'll cross the x-axis at -1 and 3, and because the x^2 term is negative, it'll be an n-shaped curve.



This is all the information you need to make a quick sketch to help you answer the question.

4) Now solve the inequality — you want the bit where the graph is above the x-axis (as it's a >). Reading off the graph, you can see that the solution is $-1 < x < 3$.

THE BIG QUESTION

Solve the inequality $x^2 - 5x - 24 < 0$

Solve the inequality $x^2 - x - 30 \ge 0$

$x \le -5$ or $x \ge 6$
$-3 < x < 8$

Solving Equations using Graphs

You can plot graphs to find solutions (or approximate solutions) to simultaneous equations and other equations. Plot the equations you want to solve and the solution lies where the lines intersect.

Solving Simultaneous Equations

See p.39 for more on simultaneous equations.

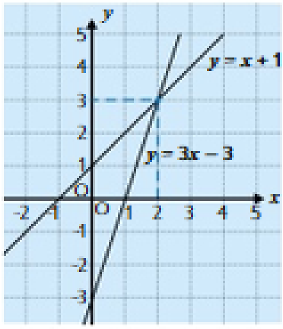
If you want to solve a pair of simultaneous equations with a graph, it's just a matter of plotting them both on a graph and writing down where they cross.

EXAMPLES:

1. Use the graph to the right to solve the simultaneous equations $y = 3x - 3$ and $y = x + 1$.

   Read off the x and y values where the two lines intersect.

   $x = 2, y = 3$



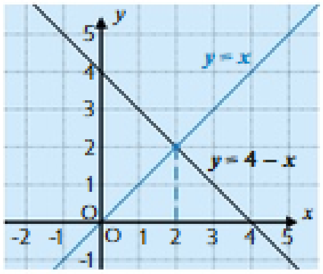
2. The graph of $y = 4 - x$ is shown to the right. Use the graph to find the solution to $4 - x = x$.

   Each side of the equation $4 - x = x$ represents a line. These lines are $y = 4 - x$ and $y = x$.

   Draw the line $y = x$ on the graph, then read off the x-coordinate where it crosses $y = 4 - x$.

   The solution is $x = 2$.

   At the point where the lines cross, both sides of the equation are equal, so this is the solution.



Solving Quadratic Equations

EXAMPLE:

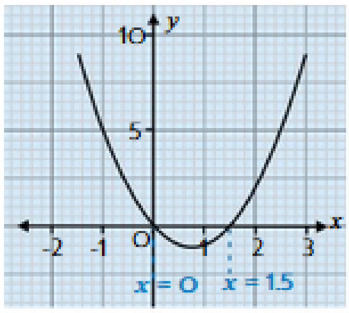
Use the graph of $y = 2x^2 - 3x$ (on the right) to find both roots of the equation $2x^2 - 3x = 0$.

The left-hand side of the equation $2x^2 - 3x = 0$ represents the curve $y = 2x^2 - 3x$, and the right-hand side represents the line $y = 0$ (the x-axis).

Read off the x-values where the curve crosses the x-axis — these are the solutions or roots.

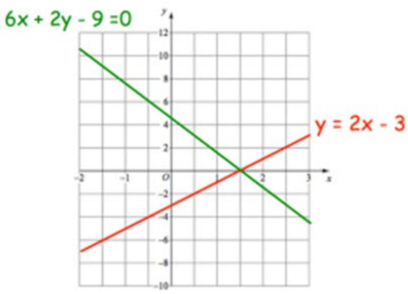
The roots are $x = 0$ and $x = 1.5$.

Quadratic equations usually have 2 roots (see p.38).



Use the graphs to solve the simultaneous equations

$6x + 2y - 9 = 0$
$y = 2x - 3$.



THE BIG QUESTION

x = 1.5, y = 0

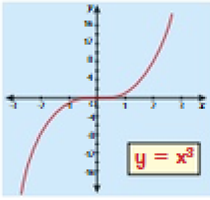
Harder Graphs

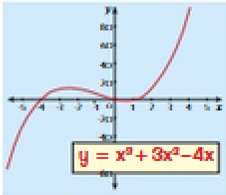
Graphs come in all sorts of shapes, sizes and wiggles — here are the first of 7 more types you need to know:

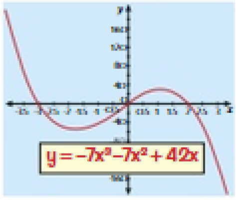
x^3 Graphs: $y = ax^3 + bx^2 + cx + d$ (b, c and d can be zero)

All x^3 graphs (also known as cubic graphs) have a wiggle in the middle — sometimes it's a flat wiggle, sometimes it's more pronounced. $-x^3$ graphs always go down from top left, $+x^3$ ones go up from bottom left.

Note that x^3 must be the highest power and there must be no other bits like $1/x$ etc.





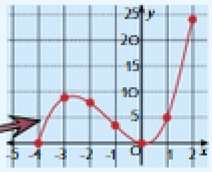


EXAMPLE: Draw the graph of $y = x^3 + 4x^2$ for values of x between -4 and $+2$.

Start by making a table of values.

x	-4	-3	-2	-1	0	1	2
$y = x^3 + 4x^2$	0	9	8	3	0	5	24

Plot the points and join them with a lovely smooth curve. DON'T use your ruler — that would be a trifle daft.



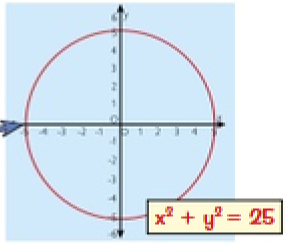
Circles: $x^2 + y^2 = r^2$ H

The equation for a circle with centre (0, 0) and radius r is:

$$x^2 + y^2 = r^2$$

$x^2 + y^2 = 25$ is a circle with centre (0, 0).
$r^2 = 25$, so the radius, r, is 5.

$x^2 + y^2 = 100$ is a circle with centre (0, 0).
$r^2 = 100$, so the radius, r, is 10.



EXAMPLE: Find the equation of the tangent to $x^2 + y^2 = 100$ at the point $(8, -6)$.

1) Find the gradient of the line from the origin to $(8, -6)$. This is a radius of the circle.

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{-6-0}{8-0} = \frac{-3}{4}$$

2) A tangent meets a radius at 90°, (see p.76) so they are perpendicular — so the gradient of the tangent is $-\frac{1}{m}$.

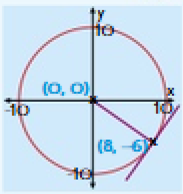
$$\text{Gradient of tangent} = -\frac{1}{m} = -\frac{1}{-\frac{3}{4}} = \frac{4}{3}$$

3) Find the equation of the tangent by substituting $(8, -6)$ into $y = mx + c$.

$$y = mx + c \Rightarrow (-6) = \frac{4}{3}(8) + c$$

$$-6 = \frac{32}{3} + c$$

$$c = -\frac{50}{3}$$



$$y = \frac{4}{3}x - \frac{50}{3}$$

THE BIG QUESTION

The equation of a circle C, with centre O, is:

$x^2 + y^2 = 225$

(a) Find the coordinates of the centre O.

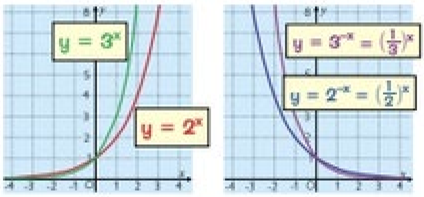
(b) Find the radius of C.

(c) Show the point (9, 12) lies on C.

a) (0, 0)
b) 15
c) $9^2 + 12^2 = 81 + 144 = 225$

Here are two more graph types you need to be able to plot or sketch. Knowing what you're aiming for really helps.

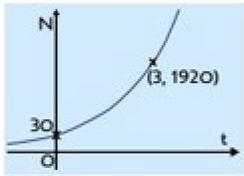
k^x Graphs: $y = k^x$ or $y = k^{-x}$ (k is some positive number) H



1) These 'exponential' graphs are always above the x-axis, and always go through the point (0, 1).
2) If k > 1 and the power is +ve, the graph curves upwards.
3) If k is between 0 and 1 OR the power is negative, then the graph is flipped horizontally.

EXAMPLE:

This graph shows how the number of victims of an alien virus (N) increases in a science fiction film. The equation of the graph is $N = fg^t$, where t is the number of days into the film. f and g are positive constants. Find the values of f and g.



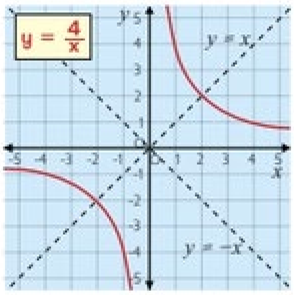
When t = 0, N = 30 so substitute these values into the equation: $30 = fg^0 \Rightarrow 30 = f \times 1 \Rightarrow f = 30$

($g^0 = 1$, so you can find f)

Substitute in t = 3, N = 1920: $N = 30g^t \Rightarrow 1920 = 30g^3$

$g = \sqrt[3]{64} \Rightarrow g = 4$

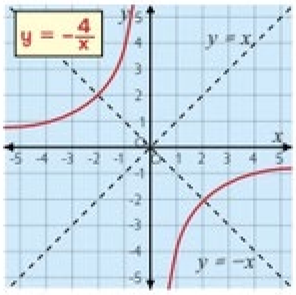
1/x (Reciprocal) Graphs: $y = A/x$ or $xy = A$ H



These are all the same basic shape, except the negative ones are in opposite quadrants to the positive ones (as shown). The two halves of the graph don't touch. The graphs don't exist for x = 0.

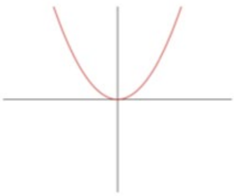
They're all symmetrical about the lines y = x and y = −x.

(You get this type of graph with inverse proportion — see p.63)



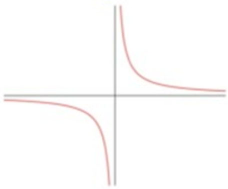
Match each graph to the correct equation

Graph A



Graph B

Graph C



Graph D

$y = x^2$ is graph **A**

$y = x^3$ is graph ....

$y = 2^x$ is graph ....

$y = \frac{1}{x}$ is graph ....

A, D, B, C

Before you leave this page, you should be able to close your eyes and picture these three graphs in your head, properly labelled and everything. If you can't, you need to learn them more. I'm not kidding.

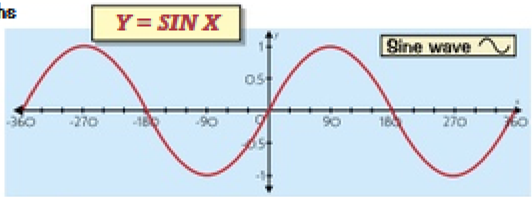
Sine 'Waves' and Cos 'Buckets' H

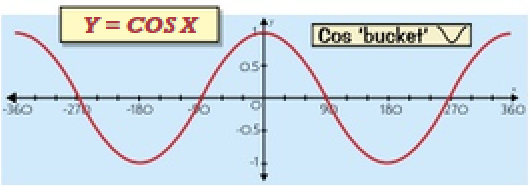
1) The underlying shape of the sin and cos graphs is identical — they both bounce between y-limits of exactly +1 and –1.

2) The only difference is that the sin graph is shifted right by 90° compared to the cos graph.

3) For 0° – 360°, the shapes you get are a Sine 'Wave' (one peak, one trough) and a Cos 'Bucket' (starts at the top, dips, and finishes at the top).

4) Sin and cos repeat every 360°. The key to drawing the extended graphs is to first draw the 0° – 360° cycle of either the Sine 'WAVE' or the Cos 'BUCKET' and then you can repeat it forever in both directions as shown above.

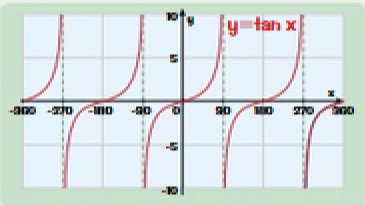




Tan x can be Any Value at all H

tan x is different from sin x or cos x — it goes between -∞ and +∞.

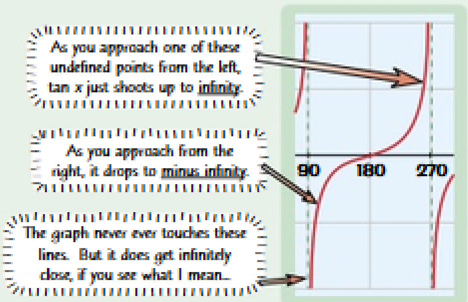
Tan x repeats every 180°



tan x goes from –∞ to +∞ every 180°.

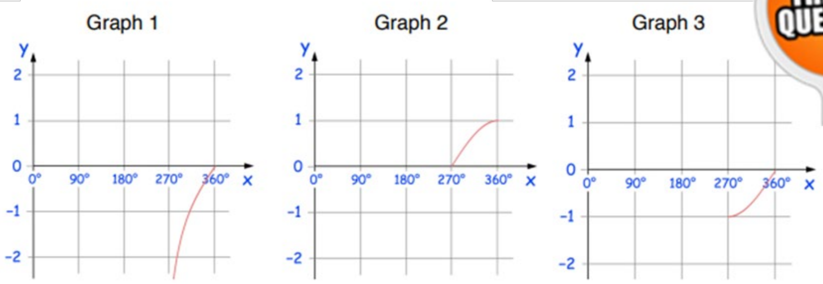
So it repeats every 180° and takes every possible value in each 180° interval.

tan x is undefined at ±90°, ±270°,...



The easiest way to sketch any of these graphs is to plot the important points which happen every 90° (e.g. –180°, –90°, 0°, 90°, 180°, 270°, 360°, 450°, 540°...) and then just join the dots up.

Here are three graphs for $270^\circ \leq x \leq 360^\circ$



(a) Which graph is $y = sin(x)$?

(b) Which graph is $y = cos(x)$?

(c) Which graph is $y = tan(x)$?

a) 3
b) 2
c) 1

Solving Equations using Graphs

You can plot graphs to find approximate solutions to simultaneous equations or other awkward equations. Plot the equations you want to solve and the solution lies where the lines intersect.

Plot Both Graphs and See Where They Cross

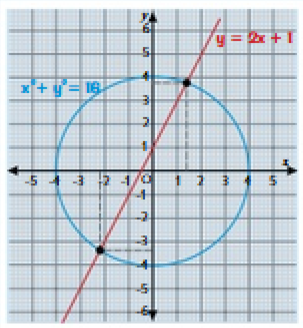
EXAMPLE:

By plotting the graphs, solve the simultaneous equations $x^2 + y^2 = 16$ and $y = 2x + 1$.

H

1) DRAW BOTH GRAPHS.
$x^2 + y^2 = 16$ is the equation of a circle with centre (0, 0) and radius 4 (see p.49). Use a pair of compasses to draw it accurately.

2) LOOK FOR WHERE THE GRAPHS CROSS.
The straight line crosses the circle at two points. Reading the x and y values of these points gives the solutions $x = 1.4$, $y = 3.8$ and $x = -2.2$, $y = -3.4$ (all to 1 decimal place).



Using Graphs to Solve Harder Equations

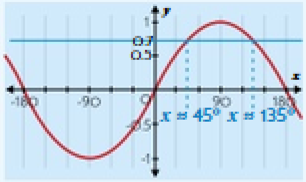
EXAMPLES:

1. The graph of $y = \sin x$ is shown to the right. Use the graph to estimate the solutions to $\sin x = 0.7$ between $-180°$ and $180°$.

H

Draw the line $y = 0.7$ on the graph, then read off where it crosses $\sin x$.

The solutions are $x \approx 45°$ and $x \approx 135°$.



2. The graph of $y = 2x^2 - 3x$ is shown on the right.

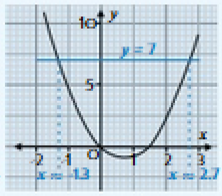
H

a) Use the graph to estimate both solutions to $2x^2 - 3x = 7$.

$2x^2 - 3x = 7$ is what you get when you put $y = 7$ into the equation:

1) Draw a line at $y = 7$.
2) Read the x-values where the curve crosses this line.
The solutions are around $x = -1.3$ and $x = 2.7$.

Quadratic equations usually have 2 solutions.



b) Find the equation of the line you would need to draw on the graph to solve $2x^2 - 5x + 1 = 0$

This is a bit nasty — the trick is to rearrange the given equation $2x^2 - 5x + 1 = 0$ so that you have $2x^2 - 3x$ (the graph) on one side.

$2x^2 - 5x + 1 = 0$

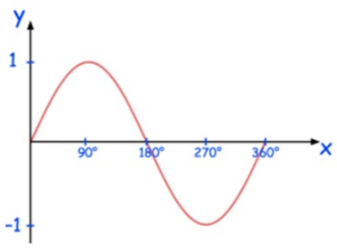
Adding $2x - 1$ to both sides: $2x^2 - 3x = 2x - 1$

So the line needed is $y = 2x - 1$.

The sides of this equation represent the two graphs $y = 2x^2 - 3x$ and $y = 2x - 1$. Finding the points where these graphs cross will give the solutions to $2x^2 - 5x + 1 = 0$

Here is the graph of $y = \sin(x)$ for $0° \le x \le 360°$

THE BIG QUESTION



One solution of $\sin(x°) = -0.5$ is $x = 210°$

(a) Find another solution of $\sin(x°) = -0.5$ for $0° \le x \le 360°$

(b) Find the solutions of $\sin(x°) = 0.5$ for $0° \le x \le 360°$

a) 330°
b) 30°, 150°

Transformations of Graphs

Don't be put off by function notation involving f(x). It doesn't mean anything complicated, it's just a fancy way of saying "an expression in x". In other words "y = f(x)" just means "y = some totally mundane expression in x, which we won't tell you, we'll just call it f(x) instead to see how many of you get in a flap about it".

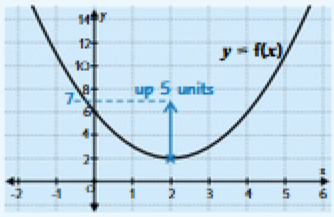
Translations on the y-axis: $y = f(x) + a$ H

You must describe this as a 'translation' in the exam — don't just say 'slide'.

This is where the whole graph is slid up or down the y-axis, and is achieved by simply adding a number onto the end of the equation: $y = f(x) + a$.

EXAMPLE:

To the right is the graph of $y = f(x)$.
Write down the coordinates of the minimum point of the graph with equation $y = f(x) + 5$.



The minimum point of $y = f(x)$ has coordinates (2, 2).
$y = f(x) + 5$ is the same shape graph, translated 5 units upwards.
So the minimum point of $y = f(x) + 5$ is at (2, 7).

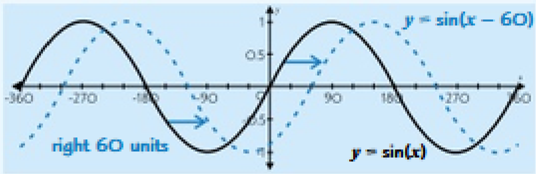
Translations on the x-axis: $y = f(x - a)$ H

This is where the whole graph slides to the left or right and it only happens when you replace 'x' everywhere in the equation with 'x – a'. These are tricky because they go 'the wrong way'. If you want to go from $y = f(x)$ to $y = f(x - a)$ you must move the whole graph a distance 'a' in the positive x-direction → (and vice versa).

EXAMPLE: The graph $y = \sin x$ is shown below, for $-360° \le x \le 360$.

a) Sketch the graph of $\sin(x - 60)°$.
$y = \sin(x - 60)°$ is $y = \sin x$ translated 60° in the positive x-direction.

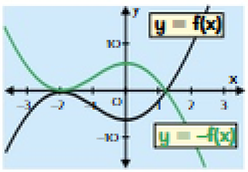
b) Give the coordinates of a point where $y = \sin(x - 60)°$ crosses the x-axis.

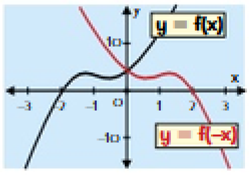


$y = \sin x$ crosses the x-axis at (0, 0),
so $y = \sin(x - 60)°$ will cross at (60°, 0)

Reflections: $y = -f(x)$ and $y = f(-x)$ H

$y = -f(x)$ is the reflection in the x-axis of $y = f(x)$.

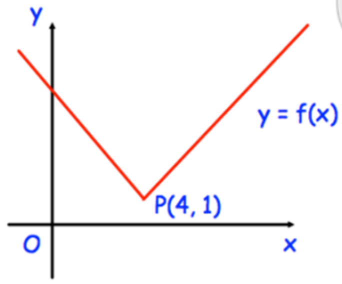




$y = f(-x)$ is the reflection in the y-axis of $y = f(x)$.

Here is the graph of y = f(x)
The point P(4, 1) is a point on the graph.

THE BIG QUESTION



What are the coordinates of the new position of P when the graph y = f(x) is transformed to the graph of

(a) $y = -f(x)$
(b) $y = f(x) + 4$
(c) $y = f(-x)$
(d) $y = f(x + 5)$

a) (4, -1)
b) (4, 5)
c) (-4, 1)
d) (-1, 1)

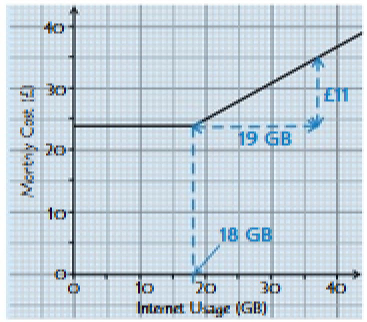
Real Life Graphs

Now and then, graphs mean something more interesting than just $y = x^3 + 4x^2 - 6x + 4$...

Graphs Can Show Billing Structures

Many bills are made up of two charges — a fixed charge and a cost per unit. E.g. You might pay £11 each month for your phone line, and then be charged 3p for each minute of calls you make.

EXAMPLE: This graph shows how a broadband bill is calculated.



a) How many gigabytes (GB) of Internet usage are included in the basic monthly cost?

18 GB

The first section of the graph is horizontal. You're charged £24 even if you don't use the Internet during the month. It's only after you've used 18 GB that the bill starts rising.

b) What is the cost for each additional gigabyte (to the nearest 1p)?

Gradient of sloped section = cost per GB

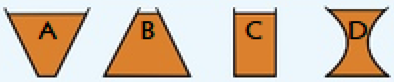
$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{11}{19} = £0.5789...$ per GB

To the nearest 1p this is £0.58

No matter what the graph, the gradient is always the y-axis unit PER the x-axis unit (see p.57).

Graphs Can Show Changes with Time H

EXAMPLE: Four different-shaped glasses containing juice are shown on the right. The juice is siphoned out of each glass at a constant rate.



Each graph below shows how the height of juice in one glass changes. Match each graph to the correct glass.

A steeper slope means that the juice height is changing faster.

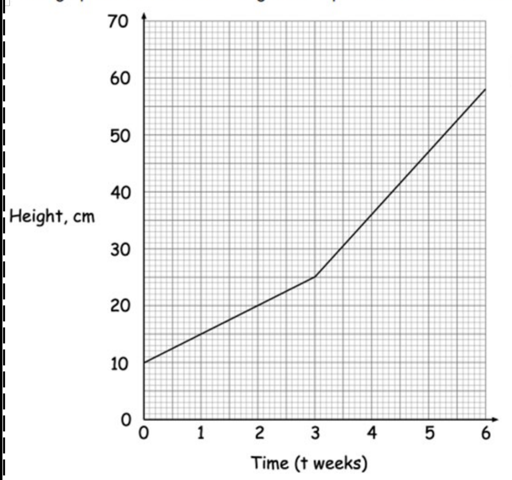
Glass C — Glass C has straight sides, so the juice height falls steadily.

Glass B — Glass B is narrowest at the top, so the juice height falls fastest at first.

Glass D — Glass D is narrowest in the middle, so the height will fall fastest in the middle part of the graph.

Glass A — Glass A is narrowest at the bottom, so the height will fall fastest at the end of the graph.

Simon has a plant that should grow at the same rate every week.
After 3 weeks, Simon starts using plant food that increases the rate of growth.
The graph below shows the height of the plant over the first 6 weeks.



THE BIG QUESTION

By how many more centimetres each week does the plant grow after giving it the plant food?

6cm per week more

Distance Time Graphs

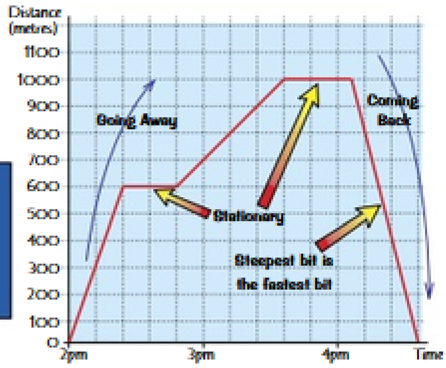
Ah, what could be better than some nice D/T graphs? OK, so a slap-up meal with Hugh Jackman might be better. Unfortunately this section isn't called 'Te... he Stars' so a D/T graph will have to do...

Distance-Time Graphs

Distance-time graphs can look a bit awkward at first, but they're not too bad once you get your head around them.

Just remember these 4 important points:

1) At any point, **GRADIENT = SPEED**.
2) The **STEEPER** the graph, the **FASTER** it's going.
3) **FLAT SECTIONS** are where it is **STOPPED**.
4) If the gradient's negative, it's **COMING BACK**.



EXAMPLE: Henry went out for a ride on his bike. After a while he got a puncture and stopped to fix it. This graph shows the first part of Henry's journey.

a) What time did Henry leave home?
He left home at the point where the line starts. At 8:15

b) How far did Henry cycle before getting a puncture?
The horizontal part of the graph is where Henry stopped. 12 km

c) What was Henry's speed before getting a puncture?
Using the speed formula is the same as finding the gradient.

$$\text{speed} = \frac{\text{distance}}{\text{time}} = \frac{12\text{ km}}{0.5\text{ hours}} = 24\text{ km/h}$$

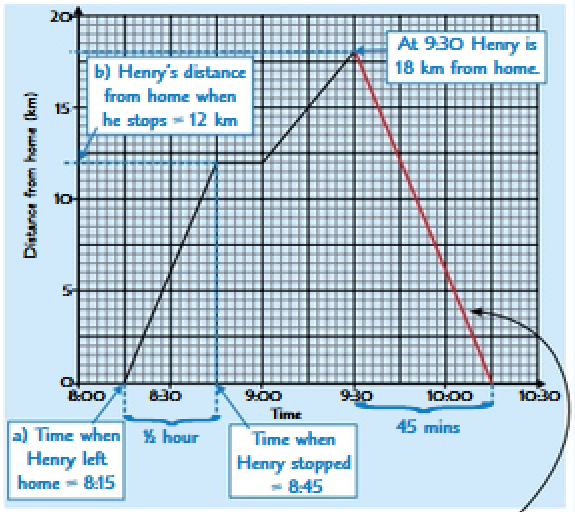
d) At 9:30 Henry turns round and cycles home at 24 km/h. Complete the graph to show this.
You have to work out how long it will take Henry to cycle the 18 km home:

$$\text{time} = \frac{\text{distance}}{\text{speed}} = \frac{18\text{ km}}{24\text{ km/h}} = 0.75\text{ hours}$$

0.75×60 mins $= 45$ mins

Decimal times are yuck, so convert it to minutes.

45 minutes after 9:30 is 10:15, so that's the time Henry gets home. Now you can complete the graph.



THE BIG QUESTION

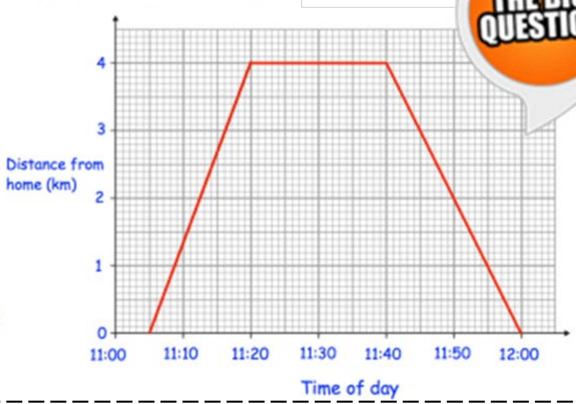
Laura goes for a cycle from her house to the post office, 4km away.

(a) How long did it take Laura to cycle to the post office?

(b) Work out Laura's speed cycling to the post office.

(c) How long did Laura spend at the post office?

(d) Work out Laura's speed cycling back home.



a) 15 mins
b) 16km/h
c) 20 mins
d) 12km/h

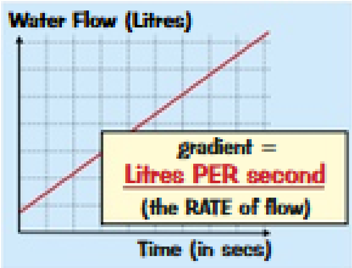
Gradient of Real Life Graphs

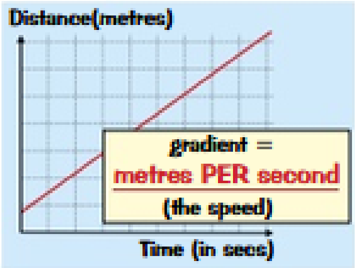
Gradients are great — they tell you all sorts of stuff, like 'you're accelerating', or 'you need a spirit level'.

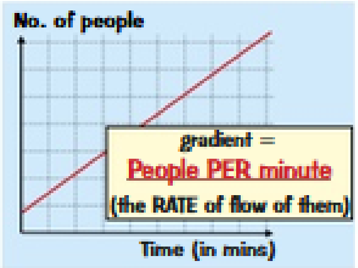
The Gradient of a Graph Represents the Rate

No matter what the graph may be, the meaning of the gradient is always simply:

(y-axis UNITS) PER (x-axis UNITS)







Finding the Average Gradient H

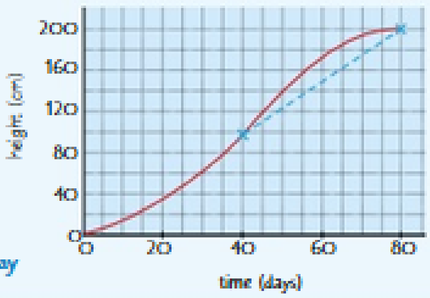
You could be asked to find the average gradient between two points on a curve.

EXAMPLE:

Vicky is growing a sunflower. She records its height each day and uses this to draw the graph shown. What is the average growth per day between days 40 and 80?

1) Draw a straight line connecting the points.
2) Find the gradient of the straight line.

Gradient $= \frac{\text{change in } y}{\text{change in } x} = \frac{200 - 100}{80 - 40} = \frac{100}{40} = 2.5$ cm per day



Estimating the Rate at a Given Point H

To estimate the rate at a single point on a curve, draw a tangent that touches the curve at that point. The gradient of the tangent is the same as the rate at the chosen point.

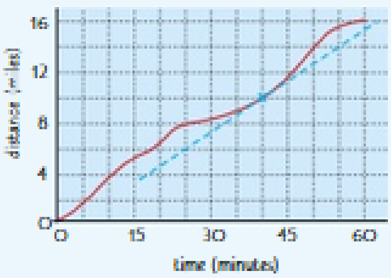
EXAMPLE:

Dan plots a graph to show the distance he travelled during a bike race. Estimate Dan's speed after 40 minutes.

1) Draw a tangent to the curve at 40 minutes.
2) Find the gradient of the straight line.

Gradient $= \frac{\text{change in } y}{\text{change in } x} = \frac{14 - 10}{55 - 40} = \frac{4}{15}$ miles per minute

$= 16$ miles per hour



THE BIG QUESTION

a) On the sunflower height graph, estimate the rate of growth on day 20.
b) On the cycling graph, calculate the average speed between 25 and 40 minutes.

a) 2.2cm per day
b) 8mph

Velocity Time Graphs

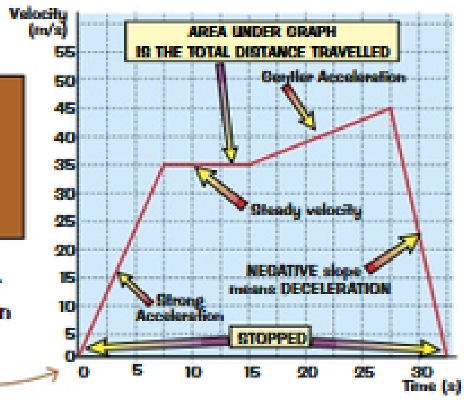
Velocity is speed measured in a particular direction. So two objects with velocities of 20 m/s and –20 m/s are moving at the same speed but in opposite directions. For the purpose of these graphs, velocity is just speed.

Velocity-Time Graphs H

1) At any point, GRADIENT = ACCELERATION.
2) NEGATIVE SLOPE is DECELERATION (slowing down).
3) FLAT SECTIONS are STEADY VELOCITY.
4) AREA UNDER GRAPH = DISTANCE TRAVELLED.

The units of acceleration equal the velocity units per time units. For velocity in m/s and time in seconds the units of acceleration are m/s per s — this is written as m/s^2.

Be careful not to get the velocity and distance-time graphs mixed up — always check the axes.



Estimating the Area Under a Curve H

It's easy to find the area under a velocity-time graph if it's made up of straight lines — just split it up into triangles, rectangles and trapeziums and use the area formulas (see p.82).

To estimate the area under a curved graph, divide the area under the graph approximately into trapeziums, then find the area of each trapezium and add them all together.

EXAMPLE:

The red graph shows part of Rudolph the super-rabbit's morning run. Estimate the distance he ran during the 24 seconds shown.

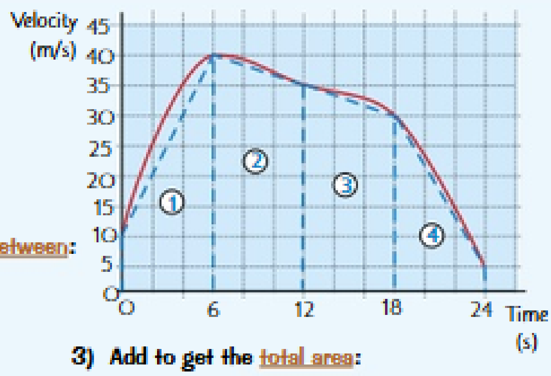
1) Divide the area under the graph into trapeziums of equal width.
2) Find the area of each using area = average of parallel sides × distance between:

Area of trap. 1 = $\frac{1}{2} \times (10 + 40) \times 6 = 150$

Area of trap. 2 = $\frac{1}{2} \times (40 + 35) \times 6 = 225$

Area of trap. 3 = $\frac{1}{2} \times (35 + 30) \times 6 = 195$

Area of trap. 4 = $\frac{1}{2} \times (30 + 5) \times 6 = 105$



3) Add to get the total area:

Total area = 150 + 225 + 195 + 105 = 675

So Rudolph ran about 675 m in total.

You could use this to estimate the average speed — just divide the total distance by the time taken.

You can find the average acceleration by finding the gradient between two points on a velocity-time curve, or estimate the acceleration at a specific point by drawing a tangent to the curve (see next page).

Calculate the total distance travelled in the velocity-time graph at the top of this page.

THE BIG QUESTION

1006.25m

Proof

I'm not going to lie — proof questions can look a bit terrifying. But there are a couple of tricks you can use that makes them a bit less scary.

Prove Statements are True or False

1) The most straightforward proofs are ones where you're given a statement and asked if it's true or false.
2) To show that it's false, all you have to do is find one example that doesn't work.
3) Showing that something is true is a bit trickier — you might have to do a bit of rearranging to show that two things are equal, or show that one thing is a multiple of a certain number.

EXAMPLE: Find an example to show that the statement below is not correct.
"The difference between two prime numbers is always even."

2 and 5 are both prime, so try them:
$5 - 2 = 3$, which is odd — so the statement is not correct.

It was easy to find an example for this one — but sometimes you might have to try a few different numbers to find a pair that doesn't work.

EXAMPLE: Prove that $(n + 3)^2 - (n - 2)^2 \equiv 5(2n + 1)$.

Take one side of the equation and play about with it until you get the other side.

$$\text{LHS: } (n + 3)^2 - (n - 2)^2 \equiv n^2 + 6n + 9 - (n^2 - 4n + 4)$$
$$\equiv n^2 + 6n + 9 - n^2 + 4n - 4$$
$$\equiv 10n + 5$$
$$\equiv 5(2n + 1) = \text{RHS} ✓$$

See p.28 for a reminder on factorising.

$\equiv$ is the identity symbol, and means that two things are identically equal to each other. So $a + b \equiv b + a$ is true for all values of a and b (unlike an equation, which is only true for certain values).

Show that One Thing is a Multiple of Another

1) To show that one thing is a multiple of a particular number (let's say 5), you need to rearrange the thing you're given to get it into the form 5 × a whole number, which means it's a multiple of 5.
2) If it can't be written as 5 × a whole number, then it's not a multiple of 5.

EXAMPLE: $a = 3(b + 9) + 5(b - 2) + 3$.
Show that a is a multiple of 4 for any whole number value of b.

$a = 3(b + 9) + 5(b - 2) + 3$
$= 3b + 27 + 5b - 10 + 3$ — Expand the brackets...
$= 8b + 20$ — ...simplify...
$= 4(2b + 5)$ — ...and factorise.

$2b + 5$ is a whole number because b is a whole number.

a can be written as $4 \times$ something (where the something is $2b + 5$) so it is a multiple of 4.

3) It's always a good idea to keep in mind what you're aiming for — here, you're trying to write the expression for a as '4 × a whole number', so you'll need to take out a factor of 4 at some point.

THE BIG QUESTION

The first two terms of a fibonacci sequence are ***a*** and ***b***.

(a) Show the 4th term of the sequence is ***a+2b***
(b) Prove that the sum of the first 10 terms is equal to 11 times the 7th term.

a, b, a+b, a+2b, 2a+3b
Both equal to 55a + 88b

I'm not going to lie — proof questions can look a bit terrifying. There are all sorts of things you could be asked to prove — I'll start with some algebraic proofs on this page, then move on to wild and wonderful topics.

Show Things Are Odd, Even or Multiples by Rearranging

Before you get started, there are a few things you need to know — they'll come in very handy when you're trying to prove things.

- Any even number can be written as $2n$ — i.e. $2 \times$ something.
- Any odd number can be written as $2n + 1$ — i.e. $2 \times$ something $+ 1$.
- Consecutive numbers can be written as n, $n + 1$, $n + 2$ etc. — you can apply this to e.g. consecutive even numbers too (they'd be written as $2n$, $2n + 2$, $2n + 4$). (In all of these statements, n is just any integer.)
- The sum, difference and product of integers is always an integer.

This can be extended to multiples of other numbers too — e.g. to prove that something is a multiple of 5, show that it can be written as $5 \times$ something.

EXAMPLE: Prove that the sum of any three odd numbers is odd.

So what you're trying to do here is show that the sum of three odd numbers can be written as $(2 \times \text{integer}) + 1$.

Take three odd numbers:
$2a + 1$, $2b + 1$ and $2c + 1$
(they don't have to be consecutive)

Add them together:

$$2a + 1 + 2b + 1 + 2c + 1 = 2a + 2b + 2c + 2 + 1$$
$$= 2(a + b + c + 1) + 1$$
$$= 2n + 1 \text{ where } n \text{ is an integer } (a + b + c + 1)$$

You'll see why I've written 3 as 2 + 1 in a second.

So the sum of any three odd numbers is odd.

EXAMPLE: Prove that $(n + 3)^2 - (n - 2)^2 \equiv 5(2n + 1)$.

Take one side of the equation and play about with it until you get the other side:

$$\text{LHS: } (n + 3)^2 - (n - 2)^2 \equiv n^2 + 6n + 9 - (n^2 - 4n + 4)$$
$$\equiv n^2 + 6n + 9 - n^2 + 4n - 4$$
$$\equiv 10n + 5$$
$$\equiv 5(2n + 1) = \text{RHS} ✓$$

$\equiv$ is the identity symbol, and means that two things are identically equal to each other. So $a + b \equiv b + a$ is true for all values of a and b (unlike an equation, which is only true for certain values).

Disprove Things by Finding a Counter Example

If you're asked to prove a statement isn't true, all you have to do is find one example that the statement doesn't work for — this is known as disproof by counter example.

EXAMPLE: Ross says "the difference between any two consecutive square numbers is always a prime number". Prove that Ross is wrong.

Just keep trying pairs of consecutive square numbers (e.g. 1^2 and 2^2) until you find one that doesn't work:

1 and 4 — difference = 3 (a prime number)
4 and 9 — difference = 5 (a prime number)
9 and 16 — difference = 7 (a prime number)
16 and 25 — difference = 9 (NOT a prime number) so Ross is wrong.

You don't have to go through loads of examples if you can spot one that's wrong straightaway — you could go straight to 16 and 25.

THE BIG QUESTION

Prove the following

(a) $(n + 4)^2 - (n + 2)^2$ is always a multiple of 4 for all positive integer values of n.

(b) Prove the product of two even consecutive numbers is always a multiple of 4.

(c) The difference between the squares of any two consecutive integers is equal to the sum of the two integers.

a) $4n + 12 = 4(n+3)$
b) $4n^2 + 4 = 4(n^2 + 1)$
c) $n^2+2n+1-n^2 = 2n + 1 = n + n + 1$

There's no set method for proof questions — you have to think about all the things you're told in the question (or that you know from other areas of maths) and jiggle them around until you've come up with a proof.

Proofs Will Test You On Other Areas of Maths

You could get asked just about anything in a proof question, from power laws...

EXAMPLE: Show that the difference between 10^{18} and 6^{21} is a multiple of 2.

$10^{18} - 6^{21} = (10 \times 10^{17}) - (6 \times 6^{20})$

$= (2 \times 5 \times 10^{17}) - (2 \times 3 \times 6^{20}) = 2[(5 \times 10^{17}) - (3 \times 6^{20})]$

which can be written as $2x$ where $x = [(5 \times 10^{17}) - (3 \times 6^{20})]$ so is a multiple of 2.

... to questions on mean, median, mode or range (see p.116)...

EXAMPLE: The range of a set of positive numbers is 5. Each number in the set is doubled. Show that the range of the new set of numbers also doubles.

Let the smallest value in the first set of numbers be n.
Then the largest value in this set is $n + 5$ (as the range for this set is 5).
When the numbers are doubled, the smallest value in the new set is $2n$
and the largest value is $2(n + 5) = 2n + 10$.
To find the new range, subtract the smallest value from the largest:
$2n + 10 - 2n = 10 = 2 \times 5$, which is double the original range.

... or ones where you have to use inequalities (see p.33-34)...

EXAMPLE: Ellie says, "If $x > y$, then $x^2 > y^2$". Is she correct? Explain your answer.

Try some different values for x and y:
$x = 2, y = 1$: $x > y$ and $x^2 = 4 > 1 = y^2$
$x = 5, y = 2$: $x > y$ and $x^2 = 25 > 4 = y^2$

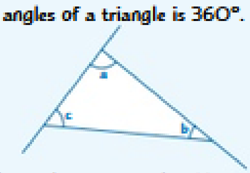
This is an example of finding a counter example — see previous page.

At first glance, Ellie seems to be correct. BUT... $x = -1, y = -2$: $x > y$ but $x^2 = 1 < 4 = y^2$, so Ellie is wrong as the statement does not hold for all values of x and y.

... or even geometric proofs (see section 5 for more on geometry).

EXAMPLE: Prove that the sum of the exterior angles of a triangle is 360°.

First sketch a triangle with angles a, b and c

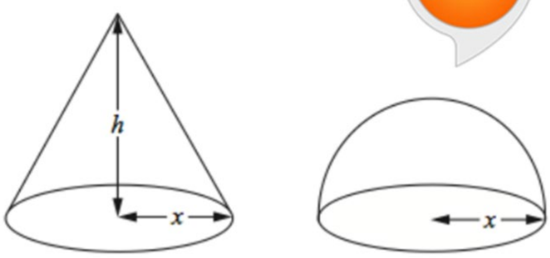


Then the exterior angles are:
180° – a, 180° – b and 180° – c

So their sum is:
(180° – a) + (180° – b) + (180° – c)
= 540° – (a + b + c) = 540° – 180° (as the angles in a triangle add up to 180°)
= 360°

THE BIG QUESTION

The diagram shows a cone and a hemisphere.



The hemisphere has base radius x cm.
The cone has base radius x cm and perpendicular height h cm.

The volume of the cone is equal to the volume of the hemisphere.

Show that $h = 2x$

$\frac{1}{3}\pi x^2 h = \frac{4}{3}\pi x^3 \div 2$

$\pi x^2 h = 2\pi x^3$

$x^2 h = 2x^3$

$h = 2x$

Algebraic Fractions

Unfortunately, fractions aren't limited to numbers — you can get algebraic fractions too.
Fortunately, everything you learnt about fractions on p.5-6 can be applied to algebraic fractions as well.

Simplifying Algebraic Fractions (H)

You can simplify algebraic fractions by cancelling terms on the top and bottom — just deal with each letter individually and cancel as much as you can. You might have to factorise first (see pages 19 and 25-26).

EXAMPLES:

1. Simplify $\frac{21x^3y^2}{14xy^3}$

÷7 on the top and bottom
÷x on the top and bottom to leave x^2 on the top
÷y^2 on the top and bottom to leave y on the bottom

$$\frac{21x^3y^2}{14xy^3} = \frac{3x^2}{2y}$$

2. Simplify $\frac{x^2 - 16}{x^2 + 2x - 8}$

Factorise the top using D.O.T.S.
Factorise the quadratic on the bottom

$$\frac{(x+4)(x-4)}{(x-2)(x+4)} = \frac{x-4}{x-2}$$

Then cancel the common factor of $(x + 4)$

Multiplying/Dividing Algebraic Fractions (H)

1) To multiply two fractions, just multiply tops and bottoms separately.
2) To divide, turn the second fraction upside down then multiply.

EXAMPLE: Simplify $\frac{x^2-4}{x^2+x-12} \div \frac{2x+4}{x^2-3x}$

Turn the second fraction upside down — Factorise and cancel — Multiply tops and bottoms

$$\frac{x^2-4}{x^2+x-12} \div \frac{2x+4}{x^2-3x} = \frac{x^2-4}{x^2+x-12} \times \frac{x^2-3x}{2x+4} = \frac{(x+2)(x-2)}{(x+4)(x-3)} \times \frac{x(x-3)}{2(x+2)} = \frac{x-2}{x+4} \times \frac{x}{2} = \frac{x(x-2)}{2(x+4)}$$

Adding/Subtracting Algebraic Fractions (H)

For the common denominator, find something both denominators divide into.

Adding or subtracting is a bit more difficult:
1) Work out the common denominator (see p.6).
2) Multiply top and bottom of each fraction by whatever gives you the common denominator.
3) Add or subtract the numerators only.

Fractions	$\frac{1}{x} + \frac{1}{3x}$	$\frac{1}{x+1} + \frac{1}{x-2}$	$\frac{1}{x} + \frac{1}{x(x+1)}$
Common denominator	$3x$	$(x+1)(x-2)$	$x(x+1)$

EXAMPLE: Write $\frac{3}{(x+3)} + \frac{1}{(x-2)}$ as a single fraction.

1st fraction: × top & bottom by $(x - 2)$
2nd fraction: × top & bottom by $(x + 3)$

$$\frac{3}{(x+3)} + \frac{1}{(x-2)} = \frac{3(x-2)}{(x+3)(x-2)} + \frac{(x+3)}{(x+3)(x-2)}$$

Common denominator will be $(x + 3)(x - 2)$

Add the numerators

$$= \frac{3x-6}{(x+3)(x-2)} + \frac{x+3}{(x+3)(x-2)} = \frac{4x-3}{(x+3)(x-2)}$$

THE BIG QUESTION

Simplify

$$\frac{y^2 - 6y}{8} \times \frac{12}{y^2 - 4y - 12}$$

$\frac{3y}{2(y+2)}$

Simultaneous Equations

2 Seven Steps for TRICKY Simultaneous Equations H

EXAMPLE: Solve these two equations simultaneously: $7x + y = 1$ and $2x^2 - y = 3$

1. Rearrange the quadratic equation so that you have the non-quadratic unknown on its own. Label the two equations ① and ②.

$7x + y = 1$ — ① $y = 2x^2 - 3$ — ②

> You could also rearrange the linear equation and substitute it into the quadratic.

2. Substitute the quadratic expression into the other equation. You'll get another equation — label it ③.

$7x + y = 1$ — ①
$y = 2x^2 - 3$ — ② ⇒ $7x + (2x^2 - 3) = 1$ — ③

Put the expression for y into equation ① in place of y.

3. Rearrange to get a quadratic equation. And guess what... You've got to solve it.

$2x^2 + 7x - 4 = 0$
$(2x - 1)(x + 4) = 0$
So $2x - 1 = 0$ OR $x + 4 = 0$
$x = 0.5$ OR $x = -4$

> Remember — if it won't factorise, you can either use the formula or complete the square. Have a look at p.27-29 for more details.

4. Stick the first value back in one of the original equations (pick the easy one).

① $7x + y = 1$
Substitute in $x = 0.5$: $3.5 + y = 1$, so $y = 1 - 3.5 = -2.5$

5. Stick the second value back in the same original equation (the easy one again).

① $7x + y = 1$
Substitute in $x = -4$: $-28 + y = 1$, so $y = 1 + 28 = 29$

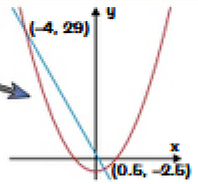
6. Substitute both pairs of answers back into the other original equation to check they work.

② $y = 2x^2 - 3$
Substitute in $x = 0.5$: $y = (2 \times 0.25) - 3 = -2.5$ — jolly good.
Substitute in $x = -4$: $y = (2 \times 16) - 3 = 29$ — smashing.

7. Write the pairs of answers out again, clearly, at the bottom of your working.

The two pairs of solutions are: $x = 0.5, y = -2.5$ and $x = -4, y = 29$

The solutions to simultaneous equations are actually the coordinates of the points where the graphs of the equations cross — so in this example, the graphs of $7x + y = 1$ and $2x^2 - y = 3$ will cross at (0.5, –2.5) and (–4, 29). There's more on this on p.52.



Find the coordinates where the line x + y = 3 and the curve $x^2 + 3y = 27$ intersect

(6, -3) and (-3, 6)

Functions

A function takes an input, processes it and outputs a value. There are two main ways of writing a function: f(x) = 5x + 2 or f: x → 5x + 2. Both of these say 'the function f takes a value for x, multiplies it by 5 and adds 2. Functions can look a bit scary-mathsy, but they're just like equations but with y replaced by f(x).

Evaluating Functions H

This is easy — just shove the numbers into the function and you're away.

EXAMPLE: $f(x) = x^2 - x + 7$. Find a) f(3) and b) f(−2)

a) $f(3) = (3)^2 - (3) + 7 = 9 - 3 + 7 = 13$ b) $f(-2) = (-2)^2 - (-2) + 7 = 4 + 2 + 7 = 13$

Combining Functions H

1) You might get a question with two functions, e.g. f(x) and g(x), combined into a single function (called a composite function).
2) Composite functions are written e.g. fg(x), which means 'do g first, then do f' — you always do the function closest to x first.
3) To find a composite function, rewrite fg(x) as f(g(x)), then replace g(x) with the expression it represents and then put this into f.

Watch out — usually $fg(x) \neq gf(x)$. Never assume that they're the same.

EXAMPLE: If $f(x) = 2x - 10$ and $g(x) = -\frac{x}{2}$, find: a) fg(x) and b) gf(x).

a) $fg(x) = f(g(x)) = f(-\frac{x}{2}) = 2(-\frac{x}{2}) - 10 = -x - 10$

b) $gf(x) = g(f(x)) = g(2x - 10) = -\left(\frac{2x - 10}{2}\right) = -(x - 5) = 5 - x$

Inverse Functions H

The inverse of a function f(x) is another function, $f^{-1}(x)$, which reverses f(x). Here's the method to find it:

1) Write out the equation x = f(y)
2) Rearrange the equation to make y the subject.
3) Finally, replace y with $f^{-1}(x)$.

f(y) is just the expression f(x), but with y's instead of x's

EXAMPLE: If $f(x) = \frac{12 + x}{3}$, find $f^{-1}(x)$.

1) Write out x = f(y): $x = \frac{12 + y}{3}$
2) Rearrange to make y the subject: $3x = 12 + y$, $y = 3x - 12$
3) Replace y with $f^{-1}(x)$: $f^{-1}(x) = 3x - 12$

So here you just rewrite the function replacing f(x) with x and x with y.

You can check your answer by seeing if $f^{-1}(x)$ does reverse f(x): e.g. $f(9) = \frac{21}{3} = 7$, $f^{-1}(7) = 21 - 12 = 9$

The functions $f(x)$ and $g(x)$ are given by the following:

$$f(x) = 2x + 1$$
$$g(x) = x - 5$$

Find:

(a) $fg(x)$ (b) $gf(x)$ (c) $ff(x)$ (d) $gg(x)$

a) 2x − 9
b) 2x − 4
c) 4x + 3
d) x − 10